Theoretical Foundations of Non-Convex Markets: With Applications to Electricity Market Design

DTU PES Summer School, Denmark

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Goals & Prerequisites

Goals

Understand economic foundations of market design relevant to electricity markets.

Prerequisites

A basic familiarity with mathematical optimization.

- Linear programming (LP) and LP duality
- Integer programming (IP)
- Basics of complexity theory (P, NP)

Welfare Theorems

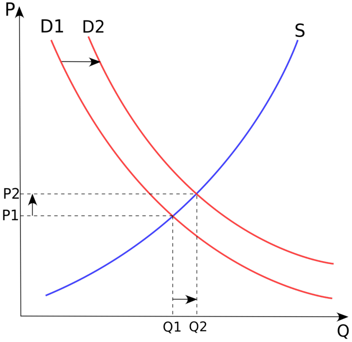


Figure: Partial equilibrium model

Theorem ((Informally) Arrow & Debreu, McKenzie 1954)

1. *The competitive equilibrium where supply equals demand is Pareto efficient.*
2. *Every Pareto efficient allocation can be supported as a competitive equilibrium.*

- All goods are divisible
- Preferences need to be convex
- Bidders are assumed to be price-takers
- Welfare theorems are existence results

Questions for the summer school

- When do markets with indivisible goods have a competitive equilibrium?
- How can we compute competitive equilibria when preferences are non-convex?

Agenda for Today

- **The Good:** Convex Markets
  - Unit Demand on Assignment Markets
  - Strong Substitutes
- **The Bad:** Non-Convex Markets
  - Linear and Non-Linear Prices
  - Approximation
- **The Ugly**: Hard Budget Constraints
  - General Valuations
  - Unit Demand

Assignment Markets (Shapley & Shubik, 1971)

Quasilinear preferences

- Payoff maximizers w. unit demand

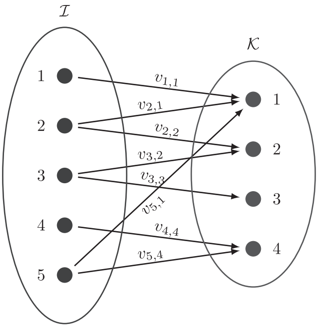
Computation

- Hungarian ($O(n^3)$)

Economics

- Welfare maximization
- Walrasian prices
- Group strategyproofness
- Ascending implementation (credibility)

Results follow from LP duality



Utilitarian Welfare and Pareto Efficiency

In a market with quasilinear (i.e., payoff-maximizing) utility maximizing aggregate surplus is a necessary condition for strong Pareto optimality. Aggregate consumers' surplus is an appropriate welfare measure in such markets (Varian, 2009).

Assignment Market: LP Formulation

Primal

$$\max \sum_{i \in \mathcal{I}} \sum_{k \in \mathcal{K}} v_{i,k} x_{i,k} \tag{1}$$

$$\begin{aligned}
\text{s.t.} \quad & \sum_{i \in \mathcal{I}} x_{i,k} = 1 && \forall k \in \mathcal{K} && (p(k)) \\
& \sum_{k \in \mathcal{K}} x_{i,k} = 1 && \forall i \in \mathcal{I} && (\pi_i) \\
& x_{i,k} \in \{0,1\} && \forall k \in \mathcal{K}, \forall i \in \mathcal{I}
\end{aligned}$$

Dual

$$\min\{\sum\nolimits_{i \in \mathcal{I}} \pi_i + \sum\nolimits_{k \in \mathcal{K}} p(k)\} \tag{2}$$

$$\begin{aligned}
\text{s.t.} \quad & \pi_i + p(k) \geq v_{i,k} && \forall k \in \mathcal{K}, \forall i \in \mathcal{I} \\
& \pi_i, p(k) \text{ unrestricted} && \forall k \in \mathcal{K}, \forall i \in \mathcal{I}
\end{aligned}$$

Dual complementary slackness: every winner maximizes payoff!

$$x_{i,k}(\pi_i - v_{i,k} + p(k)) = 0 \quad \forall k \in \mathcal{K}, \forall i \in \mathcal{I} \tag{3}$$

The Hungarian Algorithm as an Auction (Demange et al., 1986)

Table: Example of an ascending auction market. The boxed values indicate those items for which payoff is maximized at the current prices

$t = 0$	$\mathcal{K}$		
$\mathcal{I}$	A	B	C
1	5	[7]	2*
2	8*	[9]	3
3	2	[6]*	0
$p^1(k)$	0	1	0

$t = 1$	$\mathcal{K}$		
$\mathcal{I}$	A	B	C
1	5	[6]	2
2	[8]	[8]	3
3	2	[5]	0
$p^2(k)$	0	2	0

The Hungarian Algorithm as an Auction

Table: Example of an ascending auction market (contd.)

$t = 2$	$\mathcal{K}$		
$\mathcal{I}$	A	B	C
1	[5]	[5]	2
2	[8]	7	3
3	2	[4]	0
$p^3(k)$	1	3	0

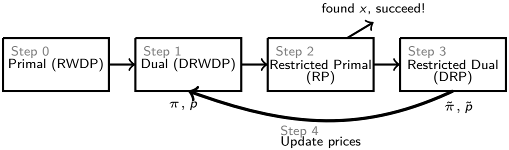
$t = 3$	$\mathcal{K}$		
$\mathcal{I}$	A	B	C
1	[4]	[4]	2
2	[7]	6	3
3	1	[3]	0
$p^4(k)$	2	4	0

The Hungarian Algorithm as an Auction

Table: Example of an ascending auction market (contd.)

$t=4$	$\mathcal{K}$		
$\mathcal{I}$	A	B	C
1	$\boxed{3}$	$\boxed{3}$	2
2	$\boxed{6}$	5	3
3	0	$\boxed{2}$	0
$p^5(k)$	3	5	0

$t=5$	$\mathcal{K}$		
$\mathcal{I}$	A	B	C
1	$\boxed{2}$	$\boxed{2}$	$\boxed{2}'$
2	$\boxed{5}'$	4	3
3	-1	$\boxed{1}'$	0
$p^6(k)$	3	5	0



In assignment markets, the buyer-optimal Walrasian prices coincide with VCG payments (Demange, Gale, Sotomayor, 1986).

Multi-Item Demand

$$\begin{aligned}
\max \quad & \sum_{i\in\mathcal{I}}\sum_{x\in\mathcal{X}_i} v_i(x) z_i(x) && && \text{(WDP)} \\
\text{s.t.} \quad & \sum_{x\in\mathcal{X}_i} z_i(x) \leq 1 && \forall i \in \mathcal{I} && (\pi_i) \\
& \sum_{i\in\mathcal{I}}\sum_{x\in\mathcal{X}_i} x(k)\, z_i(x) \leq s(k) && \forall k \in \mathcal{K} && (p(k)) \\
& z_i(x) \in \{0,1\} && \forall i \in \mathcal{I}, \forall x \in \mathcal{X}_i &&
\end{aligned}$$

Relax $z_i(x) \in \{0,1\}$ and take the dual

$$\begin{aligned}
\min \quad & \sum_{i\in\mathcal{I}} \pi_i + \sum_{k\in\mathcal{K}} s(k) p(k) && && \text{(DRWDP)} \\
\text{s.t.} \quad & \pi_i + \sum_{k\in\mathcal{K}} x(k) p(k) \geq v_i(x) && \forall i \in \mathcal{I}, \forall x \in \mathcal{X}_i && (z_i(x)) \\
& \pi_i \geq 0 && \forall i \in \mathcal{I} && (a_i) \\
& p(k) \geq 0 && \forall k \in \mathcal{K} && (b_k)
\end{aligned}$$

Multi-Item Demand

Consider a market with three items $\mathcal{K} = \{A, B, C\}$ and two bidders with v_1 and v_2.

	$x_\emptyset$	x_A	x_B	x_C	x_{AB}	x_{AC}	x_{BC}	x_{ABC}
x	$(0,0,0)$	$(1,0,0)$	$(0,1,0)$	$(0,0,1)$	$(1,1,0)$	$(1,0,1)$	$(0,1,1)$	$(1,1,1)$
$v_1(x)$	0	1	2	1	2	2	2	2
$v_2(x)$	0	1	2	2	3	2	3	3

- The optimal solution of the RWDP: $z_1(x_B) = z_1(x_{AC}) = z_2(x_C) = z_2(x_{AB}) = 0.5$.
- The optimal value of the RWDP: 4.5.
- The optimal value of the integral solution: 4.

Definition (Walrasian Competitive equilibrium, CE)

A price vector p^* and a feasible allocation $(x_1, \ldots, x_n)$ form a *competitive equilibrium* if $\sum_{i \in \mathcal{I}} x_i = s$ and $x_i \in D_i(p^*)$ for every bidder $i \in \mathcal{I}$.

Theorem (Bikhchandani and Mamer, 1997)

A Walrasian CE exists if and only if the RWDP has an optimal integral solution.

When Are There Integral Solutions?

Definition (Favati and Tardella, 1990)

A function $f : \mathbb{Z}^m \rightarrow \mathbb{R}$ is called **integrally convex** if the local convex extension of f is convex, or **integrally concave** if the function $-f$ is integrally convex.

Gross substitutes as a *sufficient* condition on the individual value functions $v_i(S)$ to yield a concave aggregate value function (Murota, 1996; Bikhchandani & Mamer, 1997).

Definition (Gross substitutes, GS (Kelso and Crawford, 1982))

(Informally:) An agent is said to have a GS valuation if, whenever the prices of some items increase and the prices of other items remain constant, the agent's demand for the items whose prices remain constant weakly increases.

1. Different definitions exist for GS (Leme, 2017), e.g., $M^\natural$-concavity (Fujishige and Yang, 2003) or based on Minkowski differences (Baldwin, Bichler, Fichtl, Klemperer, 2022).
2. Strong substitutes (SS) extend GS to multi-item, multi-unit settings (Milgrom and Strulovici, 2009).

Computing CE with Strong Substitutes

Most authors use a subgradient algorithm on the Lyapunov function (Ausubel, 2005/2006; Murota, 2016; Sun and Yang, 2009; Baldwin et al. 2024).

Definition (Lyapunov function in market design)

The **Lyapunov function** is defined as $L(p) = \sum_{i \in \mathcal{I}} u_i(p) + \langle p, s \rangle$, where s is the supply and $u_i(p)$ is the indirect utility function of bidder $i \in \mathcal{I}$ at prices p.

Proposition (Bichler, Fichtl, Schwarz, 2020)

A vector $p^* \in \mathbb{R}^m$ minimizes the DRWDP if and only if it is a minimizer of the Lyapunov function $L(p) = \sum_{i \in \mathcal{I}} u_i(p) + \langle p, s \rangle$.

Objective of the DRWDP if CE exists: $\min \sum_{i \in \mathcal{I}} \pi_i + \sum_{k \in \mathcal{K}} s(k)p(k)$
Substitute π_i by the tight dual constraints $\pi_i = v_i(x) - \sum_{k \in \mathcal{K}} x(k)p(k)$ for

$$\min_p \sum_{i \in \mathcal{I}} \max_{x \in \mathbb{Z}^m_{\geq 0}} [v_i(x) - \sum_{k \in \mathcal{K}} x(k)p(k)] + \sum_{k \in \mathcal{K}} s(k)p(k). \tag{4}$$

- The subgradient algorithm on the Lyapunov function is equivalent to a specific primal-dual algorithm on the DRWDP in this case.

Convexity to the Rescue

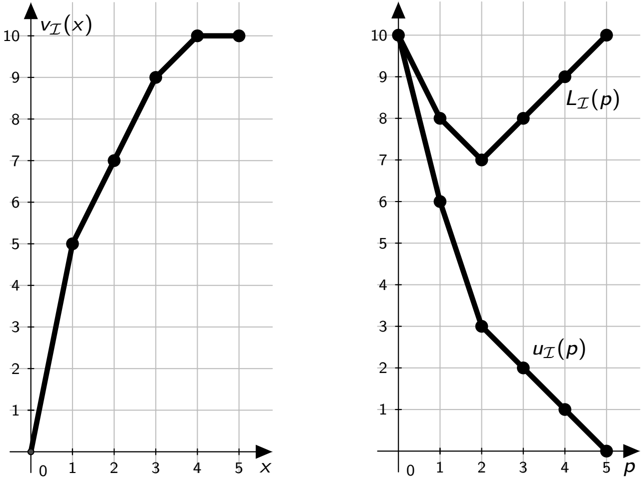
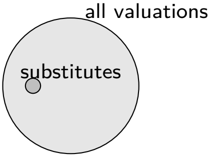


Figure: For a market with two units of a single indivisible item x, the figure shows the aggregate valuation function $v_{\mathcal{I}}(x)$, the aggregate utility function $u_{\mathcal{I}}$, and the Lyapunov function $L_{\mathcal{I}}(p)$. The Lyapunov function is minimized at $p = 2$, denoting the Walrasian equilibrium prices.

A Recipe for Convex and Quasi-Linear Markets

With quasi-linear preferences, follow the below recipe:

1. Model the allocation problem as an optimization problem
2. If the problem is convex, dual variables serve as CE prices
3. Fast iterative algorithms to compute CE are available
4. Primal-dual algorithms serve as a model for ascending auctions



Caveats:

- Incentive-compatibility and competitive equilibrium are at odds in general.
- Difficult to know a priori, if the substitutes condition is satisfied.
- Real-world markets are often not convex (=> next slides).

Agenda for Today

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  - Unit Demand on Assignment Markets
  - Strong Substitutes
- **The Bad:** Non-Convex Markets
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  - Approximation
- The Ugly: Hard Budget Constraints
  - General Valuations
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Non-Convex Preferences

1. Allocation problems on markets are typically *non-convex*
   - Examples: electricity day ahead markets with block bids or multi-part bids, transportation tenders, spectrum auctions, etc.
   - Complex constraints lead to non-convexities: block bids, volume discounts, etc.
2. We can solve very large non-convex allocation problems nowadays
   - 1 million times performance increase in IP solvers in < 30 years, but ...
3. We cannot draw on duality theory in convex optimization anymore!

Question:

- How to compute prices if participants have non-convex preferences?

Unit Commitment and Economic Dispatch Problem (UCED)

The UCED is a (non-convex) textbook-style model in electricity market design.

$$\min_{\mathbf{x},\mathbf{p}} \sum_{i\in[I]} F_i x_i + C_i p_i \tag{UCED}$$

$$\text{s.t.} \sum_{i\in[I]} p_i = \sum_{j\in[J]} \overline{q}_j, \tag{5}$$

$$p_i \le \overline{p}_i^{\max} x_i \qquad \forall i \in [I], \tag{6}$$

$$p_i \ge 0 \qquad \forall i \in [I],$$

$$x_i \in \{0,1\} \qquad \forall i \in [I].$$

- For each generator $i \in [I]$, resource i incurs a fixed cost F_i and a variable cost C_i.
- $\overline{q}_j$ is the forecast net demand at node $j \in [J]$.
- $\overline{p}_i^{\max}$ is the capacity of resource i.
- x_i is the binary commitment variable $\implies$ a *non-convex* problem.

US Bid Language

US electricity markets use *multi-part bids* which feed into fixed and variable costs of the UCED.

EU Single Day Ahead Coupling (SDAC)

EU Single Day Ahead Coupling (SDAC)[a]

- 98,6% of EU consumption is coupled
- 1.530 TWh / year coupled in one market solution
- 200 Million Euro average daily value of matched trades

Approximate size of the power grid for continental Europe and Ireland

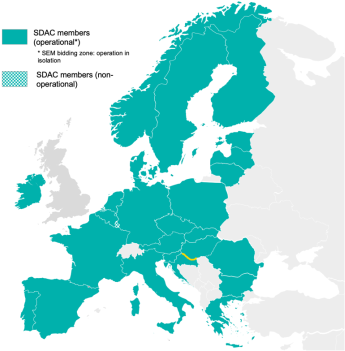
- 16,000 generators and batteries
- 25,000 nodes
- 22,000 lines

EU Bid Language

SDAC uses bids on a single MTU (15 min) and on blocks of MTUs as a bid language.

Block bids lead to a **combinatorial exchange** requiring non-convex optimization.

[a] https://www.entsoe.eu/



- The EUPHEMIA market clearing algorithm determines prices per MTU for each bidding zone once per day.

Pricing on Combinatorial Exchanges

How to find CE prices in a combinatorial exchange?

Let's start with a simple example with two items (or MTUs) only. Buyers submit bids for single items and for the block A, B, i.e. a combinatorial exchange.

	{A}	{B}	{A,B}
Buyer b_1	2	3	5
Buyer b_2	1	4	7^*

Table: 2 buyers and 1 seller

How would you set (linear) prices?
What about $p(A) = 2.5$, $p(B) = 4.5$, $p(A, B) = 7$?

Problems on Combinatorial Exchanges

	{A,B}	{B,C}	{A,C}	{A,B,C}
Buyer b_1	30	10	10	40
Buyer b_2	10	30	10	40
Buyer b_3	10	10	30	41^*

Table: 3 buyers and 1 seller

How would a single seller set prices?

- Linear (item-level) and anonymous competitive equilibrium prices are impossible e.g., $p_A + p_B \geq 30$, $p_B + p_C \geq 30$, $p_A + p_C \geq 30$ which implies $p(\{A, B, C\}) \geq 45$.
- Non-linear and personalized competitive equilibrium prices always exist with a single seller e.g., $p_2(\{B, C\}) > 30$ and $p_3(\{A, B, C\}) \leq 41$.

What if we had 3 sellers, one for each good?

Insights

With block bids and pure payoff-maximization

- CE prices need to be non-linear and personalized.
- CE prices might not always exist (the Core can be empty in a double auction).

Finding Prices in Combinatorial Exchanges

A binary program for winner determination with $X = (x_i(S))_{i,S}$ and $Y = (y_j(Z))_{j,Z}$:

$$
\begin{aligned}
\max \sum_{i \in I} \sum_{S \subseteq K} v_i(S) x_i(S) - \sum_{j \in J} \sum_{Z \subseteq K} v_j(Z) y_j(Z) & & & \\
\text{s.t.} & & & \\
\sum_{S: k \in S} \sum_{i \in I} x_i(S) - \sum_{Z: k \in Z} \sum_{j \in J} y_j(Z) & = 0 & \forall k \in K & \quad p(k) \\
\sum_{S \subseteq K} x_i(S) & \leq 1 & \forall i \in I & \quad (\pi_i) \\
\sum_{Z \subseteq K} y_j(Z) & \leq 1 & \forall j \in J & \quad (\pi_j) \\
x_i(S) & \in \{0,1\} & \forall i, S & \\
y_j(Z) & \in \{0,1\} & \forall j, Z &
\end{aligned}
$$

An extended formulation:

$$
\begin{aligned}
\max \quad & \sum_{i \in I} \sum_{S \subseteq K} v_i(S) x_i(S) - \sum_{j \in J} \sum_{Z \subseteq K} v_j(Z) y_j(Z) & & \\
\text{s.t.} \quad & x_i(S) - \sum_{x: x_i = S} \delta_{X,Y} = 0 & \forall i \in I, \forall S \subseteq K & \quad (p_i(S)) \\
& -y_j(Z) + \sum_{y: y_j = Z} \delta_{X,Y} = 0 & \forall j \in J, \forall Z \subseteq K & \quad (p_j(Z)) \\
& \sum_{S \subseteq K} x_i(S) \leq 1 & \forall i \in I & \quad (\pi_i) \\
& \sum_{Z \subseteq K} y_j(Z) \leq 1 & \forall j \in J & \quad (\pi_j) \\
& \sum_{(X,Y) \in \Gamma} \delta_{X,Y} = 1 & & \quad (\pi_a) \\
& 0 \leq x_i(S) & \forall S \subseteq K, \forall i \in I & \\
& 0 \leq y_j(Z) & \forall S \subseteq K, \forall j \in J & \\
& 0 \leq \delta_{X,Y} & \forall (X,Y) \in \Gamma &
\end{aligned}
\qquad \mathbf{P}
$$

The Extended Formulation is Integral

Lemma

*The linear program **P** (last slide) will always yield an integral solution, i.e., a feasible solution to the assignment problem.*

Proof sketch:

- The linear program **P** introduces a variable for each feasible allocation $\delta_{X,Y}$.
- The fifth constraint of **P** guarantees that at most one allocation is selected.
- The objective function ensures that the allocation maximizing gains from trade is chosen.
- The formulation has an integral optimal extreme point: in the absence of ties, exactly one $\delta_{X,Y}$ is equal to one; with ties, an integral optimal vertex can be selected.
- Because of the integrality of the δ variables, $x_i(S)$ and $y_j(Z)$ are also integral as of the first two constraints.

Competitive Equilibrium and Core Revisited

Definition

A *competitive equilibrium* for the model **P** is an assignment (x^*, y^*) with prices (p_i^*, p_j^*) such that

1. (x^*, y^*) is feasible,
2. buyers maximize utilities at these prices:
   $\forall\ i,\ v_i(S^*) - p_i^*(S^*) = \max_S \left(v_i(S) - p_i^*(S)\right)$
3. sellers maximize profits at these prices:
   $\forall\ j,\ p_j^*(Z^*) - v_j(Z^*) = \max_Z \left(p_j^*(Z) - v_j(Z)\right)$
4. the budget is balanced: $\sum_{i \in I} p_i(S^*) = \sum_{j \in J} p_j(Z^*)$

Definition

In game theory, the *core* is the set of feasible allocations that cannot be improved upon by a subset (a coalition) of the economy's agents.

Duals and Competitive Equilibrium

$$
\begin{array}{llll}
\min & \sum_{i\in I} \pi_i + \sum_{j \in J} \pi_j + \pi_a & & \\
\text{s.t.} & \pi_i \geq v_i(S) - p_i(S) & \forall i \in I, \forall S \subseteq K & (x_i(S)) \\
& \pi_j \geq p_j(Z) - v_j(Z) & \forall j \in J, \forall Z \subseteq K & (y_j(S)) \\
& \sum_{y_j(Z)\in Y} p_j(Z) - \sum_{x_i(S)\in X} p_i(S) + \pi_a \geq 0 & \forall (X,Y) \in \Gamma & (\delta_{X,Y}) \\
& \pi_i, \pi_j, p_i(S), p_j(Z) \geq 0 & \forall S, Z \subseteq K, & \\
& & \forall i \in I, \forall j \in J & \\
& \pi_a \in \mathbb{R} & &
\end{array} \qquad \textbf{(Dual)}
$$

Complementary Slackness

$$
\begin{array}{ll}
x_i(S)[\pi_i - (v_i(S) - p_i(S))] = 0 & \forall S \subseteq \mathcal{K}, \forall i \in \mathcal{I} \\
y_j(Z)[\pi_j - (p_j(Z) - v_j(Z))] = 0 & \forall Z \subseteq \mathcal{K}, \forall j \in \mathcal{J} \\
\left(\sum_{y_j(Z)\in y} p_j(Z) - \sum_{x_i(S)\in x} p_i(S) + \pi_a\right) \delta_{X,Y} = 0 & \forall (X,Y) \in \Gamma
\end{array} \qquad \text{(DCS)}
$$

The prior example with an empty core would yield fractional solutions if we enforce $\pi_a = 0$.

Budget Balance

Complementary Slackness

$$\begin{aligned} & x_i(S)[\pi_i - (v_i(S) - p_i(S))] = 0 && \forall S \subseteq \mathcal{K}, \forall i \in \mathcal{I} \\ & y_j(Z)[\pi_j - (p_j(Z) - v_j(Z))] = 0 && \forall Z \subseteq \mathcal{K}, \forall j \in \mathcal{J} \\ & \left(\sum_{y_j(Z) \in y} p_j(Z) - \sum_{x_i(S) \in x} p_i(S) + \pi_a \right) \delta_{X,Y} = 0 && \forall (X, Y) \in \Gamma \end{aligned} \quad \text{(DCS)}$$

Proposition (Bichler and Waldherr, 2018)

*The assignment (X^*, Y^*) and prices (P_i^*, P_j^*) derived from an optimal solution of **Primal** and an optimal solution of **Dual** form a competitive equilibrium for the model **Primal** if and only if $\pi_a^* = 0$.*

The prior example with an empty core would yield fractional solutions if we enforce $\pi_a = 0$.

Friedrich A. Hayek on Prices

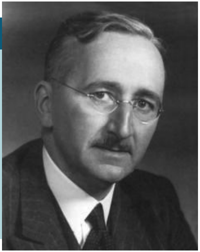
Key insights so far

With block bids only non-linear and personalized competitive equilibrium prices are possible, and even they might not always exist.

The Use of Knowledge in Society, AER, 1945

The most significant fact about this (price) system is the economy of knowledge with which it operates, or how little the individual participants need to know in order to be able to take the right action.

Friedrich A. Hayek
(Nobel Prize in Economics, 1974)



Back to Electricity Markets

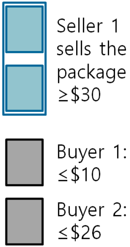
What about linear prices?

Exponentially many prices are impractical.

- Simplicity of interpretation and transparency with only a linear number of prices.
- Prices per MTU serve as baseline for derivative markets.
- No two sellers with the same package should get different payments.

Welfare Losses?

- Linear and anonymous prices can restrict the maximum gains from trade.



What Else Can Go Wrong? Paradoxically Rejected Bids

Paradoxically rejected bids	
s_1: AAA \$60	b_1: A \$30
s_2: A \$10	b_2: A \$30
	b_3: A \$30

- Efficient allocation: match s_1 with $b_1 - b_3$: Welfare 90-60=30
  - $p = 25, \pi_{s_1} = 75 - 60 = 15, \pi_{b_i} = 30 - 25 = 5$
  - Paradoxically rejected bid: s_2 has bid $10 < 25$

A Practical Issue on Electricity Markets

A frequent issue on day-ahead electricity markets: generators with low marginal costs are paradoxically rejected, while generators with higher marginal costs are dispatched.

High Efficiency (w. Linear Prices) in Large Markets

Proposition (Bichler, Fux, Goeree, 2018)

Suppose we have a multi-unit market with one seller selling a package of k units, and n unit-demand buyers, and the valuations of the buyers and the seller are drawn from the same distribution $f_Y = f_X$. The expected efficiency loss due to linear prices is less than $\frac{k-1}{n+1}$.

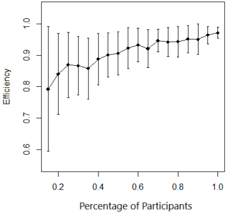


Figure: Results from numerical experiments

How to Deal with Non-Convexities?

In the EU day-ahead market:
Suboptimal outcome with a welfare loss due to linear prices. No side-payments, but paradoxically rejected bids are allowed.

In the USA real-time market:
Welfare-maximizing outcome, but the market operator pays make-whole payments and imposes penalties for deviating from the optimal dispatch.

Pricing Efficient Dispatch on (US) Electricity Markets

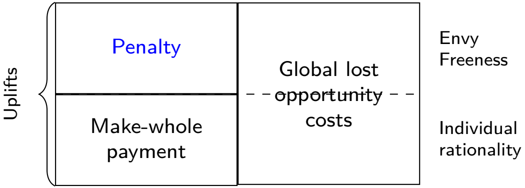
Pricing on US ISO markets:

The market operator compensates incentives to deviate:

- Convex-Hull Pricing (Hogan and Ring, 2003) and ELMP pricing (MISO, 2019) aim to *minimize GLOCs*
- IP-Pricing (O'Neill et al., 2005) aim to *minimize LLOCs*

Regulator's concerns:

- "The use of side-payments can undermine the market's ability to send actionable price signals." (U.S. FERC, 2018)
- "Convex Hull Pricing may produce positive congestion prices for transmission lines that are not congested as dispatched." (Schiro et al. 2015)



Pricing as Multi-Objective Optimization Problem

1. Solve the welfare-maximization problem: $x^* \in \arg\max_x W(x)$.
2. Solve one joint pricing optimization minimizing the max of LLOC or MWPs.

Join pricing rule (Ahunbay, Bichler, Knörr, 2024)

Minimize the maximum of make-whole payments (MWPs) and local lost opportunity costs (LLOCs).

$$\min_p \sum_i \max\left\{\lambda_i^{MWP}(p \mid x_i^*),\ \lambda_i^{LLOC}(p \mid x_i^*)\right\}$$

An experiment based on ENTSO-E bid zone review data:
(One example day in January 2009, 4538 generators, 1687 nodes)

	MWPs ($)	LLOCs ($)
Min GLOC (ELMP)	24,577	44,726
Min LLOC (IP-Pricing)	22,487	0
Min MWP	0	8,933,860
Min LLOC v MWP	326	1,292

Additional Features of Electricity Markets

- Linear approximations vs. convex relaxations of **optimal powerflow problems**
  - Optimal Powerflow Models (ACOPF) can be formulated as non-convex, non-linear optimization problem.
  - DCOPF is a linear approximation of ACOPF that ignores reactive power, voltage magnitudes, and other AC feasibility constraints.
  - In the studied instances, convex relaxations such as SOC and QC reduce redispatch, increase welfare, and lead to better prices. (Bichler, Knörr, Energy Economics, 2023).
- **Computational problems** on electricity spot markets **are very large**
  - A budget-balanced two-price markup mechanism replaces exact non-convex market clearing with convex relaxation plus rounding (Ahunbay, Bichler, Knörr, Dobos, EJOR, 2024)
- With high levels of renewables, **robust clearing and pricing** is of interest.
  - New pricing rules when robust optimization is used for market clearing (Bayrak, Bichler, Dobos, ArXiV, 2026)
- Electricity spot markets are **networked markets** on the electricity grid.
  - The Welfare Theorems can be extended to networked markets (Ahunbay, Bichler, Knörr, Operations Research, 2024)

Electricity Spot Markets As Networked Markets

Electricity spot markets are nodes on an electricity grid that are coupled via transmission lines, not a simple centralized market.

Theorem (Welfare Theorems for Coupled Markets, (Ahunbay, Bichler, Knörr, 2023))

Suppose that the cost and value functions of market participants in a coupled market are convex. Let price vector $p^* \in \mathbb{R}^{M \cup F}$ and the allocation $(z_\ell)^*_{\ell \in L}$ be a Walrasian equilibrium, then this allocation maximizes social welfare. Conversely, if $(z_\ell)^*_{\ell \in L}$ is a welfare-maximizing allocation, then it can be supported by a Walrasian price vector p^* that forms a Walrasian equilibrium.

- Proof via Fenchel-Young inequalities allowing for non-linear, but convex cost functions.
- The proof does not require differentiability of the functions.
- The model allows for fast computation via (discrete) convex optimization.

Agenda for Today

- The Good: Convex Markets
  - Unit Demand on Assignment Markets
  - Strong Substitutes
- The Bad: Non-Convex Markets
  - Linear and Non-Linear Prices
  - Approximation
- **The Ugly**: Hard Budget Constraints
  - General Valuations
  - Unit Demand

Why Budget Constraints Matter in Electricity Markets

- Electricity markets are typically modeled with **quasilinear** preferences:

$$u_b(x_b, p) = v_b(x_b) - p^\top x_b.$$

- This assumes that buyers can always pay for any welfare-improving allocation.

Motivation

Budget constraints $p^\top x_b \leq B_b$ become relevant as demand becomes more flexible and price-responsive. Examples include industrial consumers, aggregators, EV charging fleets, data centers, and local flexibility markets.

- A buyer may value electricity highly, but only up to an expenditure limit.
- High scarcity prices can make otherwise efficient demand infeasible for budget-constrained buyers.

Design question

How should electricity markets clear and price energy when buyers have willingness-to-pay **and** budget constraints?

What if Bidders Have Hard Budget Constraints?

Core

An allocation-price pair is in the core, if there is no coalition that can deviate and make all members strictly better off.

	{A}	{B}	{A,B}	Budget
Buyer b_1	0	0	10	3
Buyer b_2	2	0	2	2
Buyer b_3	0	2	2	2

Two sellers, selling individual items A, B, respectively

Assigning AB to b_1 is maximizing aggregate surplus ("welfare"), but not in the core

- b_1 can not pay 2 to each seller
- One seller and b_2 will form a blocking coalition

Core:

- b_2 gets A for a price of 2 (Payoff: 0)
- b_3 gets B for a price of 2 (Payoff: 0)
- Payoff for each seller: 2

Core and Aggregate Surplus Maximization

The surplus-maximizing outcome is not always in the core.

Strong Pareto Optimality with Budget Constraints

Utilitarian Welfare and Pareto optimality

Does maximizing aggregate surplus ("welfare") lead to SPO under hard budget constraints?

In a budget-constrained assignment market, an outcome is (μ, p) with hard budget feasibility:

$$p(\mu(i)) \leq b_i \quad \forall i, \qquad p(0) = 0, \qquad p(j) > 0 \Rightarrow \exists i : \mu(i) = j.$$

(Weak) Pareto optimality (PO).

No feasible (μ', p') such that **everyone strictly improves**:

$$v_i(\mu'(i)) - p'(\mu'(i)) > v_i(\mu(i)) - p(\mu(i)) \; \forall i, \quad p'(j) > p(j) \; \forall j.$$

(plus budget feasibility $p'(\mu'(i)) \leq b_i$)

Core $\Rightarrow$ Weak PO (well known)

Strong Pareto optimality (SPO).

No feasible (μ', p') such that **everyone weakly improves and someone strictly improves**:

$$v_i(\mu'(i)) - p'(\mu'(i)) \geq v_i(\mu(i)) - p(\mu(i)) \; \forall i, \quad p'(j) \geq p(j) \; \forall j,$$

and at least one strict inequality holds (for some buyer utility or some seller price).

Core and Strong Pareto Optimality with Budgets

Example: One indivisible item, two buyers: $v_1 = 1$, $v_2 = 100$.

1. W/o budget constraints, allocating the item to buyer 2 at $p = 1$ is SPO.
2. With budget constraints $b_1 = b_2 = 1$:

Two feasible core outcomes at price $p = 1$:

- **Allocate to buyer 1 at $p = 1$:** utilities $(u_1, u_2) = (0, 0)$.
- **Allocate to buyer 2 at $p = 1$:** utilities $(u_1, u_2) = (0, 99)$.

$$\text{Core} \Rightarrow \text{Weak PO (well known)}$$
$$\text{Core} \not\Rightarrow \text{Strong PO}$$
$$\text{Core} \wedge \text{Max. Surplus} \Rightarrow \text{Strong PO}$$

Proposition 3 (Batziou, Bichler, Fichtl, 2026).

In an assignment market with quasilinear utility functions and hard budget constraints, selecting a core allocation $\mu^\star$ at prices p that maximizes aggregate surplus among all core allocations at p is sufficient for $(\mu^\star, p)$ to be **strongly Pareto-optimal**.

The Core and Incentive-Compatibility Clash with Budget Constraints

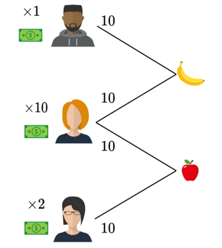
Theorem (Batziou, Bichler, Fichtl, 2026)

In assignment markets with quasilinear preferences and budget-constrained bidders with unit demand, there is no ex-post incentive-compatible deterministic mechanism terminating in the core for every input.

Proof sketch:

- Three bidders $1, 2, 3$ and two items A, B with $v_i(A) = v_i(B) = 10$ and $b^1 = b^2 = b^3 = 1$.
- If bidders report budgets truthfully, bidder 3 (wlog) has a utility of 0.
- If bidder 3 misreports $b^3 = 2$, she would receive an item and have a utility of 9.
- $p(A) = 1$ for bidder 1, and $p(B)$ for bidder 3 cannot be higher, otherwise he would envy bidder 1.
- Thus $p(A) = p(B) = 1$ and bidder 3 has a utility of 9.

An Iterative Auction Algorithm Terminating in the Core



	p^t(🍌, 🍎)				I
$t=1$	$(0,0)$	🍌	🍌🍎	🍎	$\emptyset$
$t=2$	$(1,1)$	🍌	🍌🍎	🍎	$\emptyset$
$t=3$	$(2,2)$	$\{0\}$	🍌🍎	🍎	{1
$t=4$	$(1,1)$	$\{0\}$	🍌🍎	🍎	$\emptyset$

Proposition (informally), (Batziou, Bichler, Fichtl, 2022)

If there is *no more than a single bidder reaching his budget in any round*, then it is a (weakly) dominant strategy for each bidder to submit her demand set truthfully, hence the auction is ex-post incentive-compatible and strongly Pareto optimal (SPO).

Computing a Surplus-Maximizing Core Outcome is NP-Complete

$$
\begin{aligned}
\text{maximize} \quad & \sum_{i \in \mathcal{B}} \pi_i + \sum_{j \in \mathcal{S}} \pi_j \\
\text{subject to} \quad & \pi_i = \sum_{j \in \mathcal{S}} (v_i(j) - p_j) m_i(j) && \forall i \in \mathcal{B} && (1) \\
& \pi_j = \sum_{i \in \mathcal{B}} (p(j) - r_j) m_i(j) && \forall j \in \mathcal{S} && (2) \\
& \sum_{j \in \mathcal{S}} m_i(j) \leq 1 && \forall i \in \mathcal{B} && (3) \\
& \sum_{i \in \mathcal{B}} m_i(j) \leq 1 && \forall j \in \mathcal{S} && (4) \\
& \pi_i \geq \big(v_i(j) - p(j)\big) y_i(j) && \forall i \in \mathcal{B}, j \in \mathcal{S} && (5) \\
& \pi_j \geq min(v_i(j), b^i)(1 - y_i(j)) && \forall i \in \mathcal{B}, j \in \mathcal{S} && (6) \\
& r_j m_i(j) \leq p(j) \leq min(v_i(j), b^i) m_i(j) && \forall i \in \mathcal{B}, j \in \mathcal{S} && (7) \\
& y_i(j) \in \{0, 1\} && \forall i \in \mathcal{B}, j \in \mathcal{S} && (8) \\
& m_i(j) \in \{0, 1\} && \forall i \in \mathcal{B}, j \in \mathcal{S} && (9) \\
& p(j) \geq 0 && \forall j \in \mathcal{S} && (10)
\end{aligned}
\qquad \text{(q-BC)}
$$

Theorem

Maximum Surplus Budget Constrained Stable Bipartite Matching is NP-complete.

Proof via reduction from maximum independent set.

Combinatorial Exchanges with Budget Constraints

How hard is it to compute a surplus-maximizing and core-stable outcome for combinatorial exchanges subject to budget constraints?

Theorem (Bichler, Waldherr, 2022)

Computing a surplus-maximizing core allocation in a combinatorial exchange with budget constraint is Σ_2^p-complete

Reduction from 2-quantified satisfiability, ($QSAT_2$):
Given a $n + m$ variable Boolean formula $\varphi(x, y)$ in DNF with $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ is it true that $\exists x \forall y \varphi(x, y)$?

Bilevel Integer Programming

$$\begin{aligned}
\max \quad & \textstyle\sum_{i\in I}\sum_{S\subseteq K} v_i(S)x_i(S) - \sum_{j\in J}\sum_{Z\subseteq K} v_j(Z)y_j(Z) \\
\text{s.t.} \quad & \\
& \dots \\
& \sum_{S\subseteq K} p_i(S)x_i(S) \leq B_i & \forall i \in I \\
& \sum_{S\subseteq K}\sum_{i\in I} p_i(S)x_i(S) \geq \text{what coalition } C \text{ can pay} & \forall C \subset I \\
& \dots \\
& \text{Lower-level optimization problems}
\end{aligned}$$

BIPs are also Σ_2^p-complete!

Computational Experiments

		3-stable				core-stable			
				time				time	
#buyers	#sellers	solved	stable	mean	std.err.	solved	stable	mean	std.err.
5	5	8	8	0.35	0.10	8	8	2.16	0.73
5	10	8	8	1.40	0.55	6	3	25.07	14.18
5	15	8	8	18.73	10.7	7	6	14.33	3.98
10	5	8	8	13.63	8.24	5	5	51.12	21.53
10	10	6	6	25.48	12.31	1	1	321.59	–
10	15	3	3	367.02	61.60	0	0	–	–

Table: Computational results for a variable number of buyers and sellers. For 8 instances each, the number of instances for which outcomes were calculated that are n-coalition stable and the average computational time required to solve the instances in seconds is shown.

Why Not Ignore Budget Constraints?

		capped bidding			unrestricted bidding
#buyers	#sellers	instances with different allocations	average surplus loss	std. error surplus loss	instances leading to losses
5	5	2	35.48 %	3.62 %	5
5	10	1	37.33 %	2.20 %	5
5	15	0	36.54 %	2.29 %	0
10	5	6	17.52 %	3.70 %	7
	in total	9	32.17 %	2.02 %	17

Table: Negative effects of ignoring financial constraints: Surplus loss in case of capped bidding; instances with prices leading to losses in case of unrestricted bidding

Take Home Messages

Take Home Messages

- A large part of economic theory focuses on the **“good”** cases (e.g., substitutes).
- Most real-world markets have complex constraints (complementarities, etc.) leading to non-convex (**“bad”**) optimization problems.
- Relaxed notions of welfare maximization and competitive equilibrium might work well in practice (maybe **“not so bad”**).
- For general valuations and hard budget constraints, markets are intractable even for small instances. For broad classes of general valuations and hard budget constraints, no general efficient approximation guarantees are known (**“ugly”**).

Beyond Fixed Points and Existence Results

Computation is fundamental and defines what can be achieved with market design!

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Broad Policy Papers on Electricity Market Design

- M. Ș. Ahunbay, A. Ashour Novirdoust, R. Bhuiyan, M. Bichler, S. Bindu, E. Bjørndal, M. Bjørndal, H. U. Buhl, J. P. Chaves-Ávila, H. Gerard, S. Gross, L. Hanny, J. Knörr, C. S. Köhnen, L. Marques, A. Monti, K. Neuhoff, C. Neumann, E. Ocenic, M. Ott, M. Pichlmeier, J. C. Richstein, M. Rinck, F. Röhrich, P. M. Röhrig, A. Sauer, J. Strüker, M. Troncia, J. Wagner, M. Weibelzahl, and P. Zilke. Electricity market design 2030-2050: Shaping future electricity markets for a climate-neutral europe. 2022. [`https://doi.org/10.24406/fit-n-644366`]
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Software and Data

Allocation and Pricing on Electricity Markets (APEM)

APEM is a framework for electricity-market clearing, pricing, and analysis. It brings together optimization-based market models, pricing methods, network-aware post-processing, and supporting evaluation tools in a single codebase.

- Github: `https://github.com/TUM-DSS/APEM`
- Documentation: `https://tum-dss.github.io/APEM/`
- Zenodo archive: `https://doi.org/10.5281/zenodo.20424894`