DTU Summer School on Modern Optimization in Energy Systems

Conic Optimization: Part I

Pascal Van Hentenryck

Goals of the Lecture

- Conic Optimization
  - intuition
  - second-order cone programming

What is an Linear Program?

$$
\begin{aligned}
&\min c_1x_1 + \ldots + c_nx_n \\
&\text{subject to} \\
&\quad a_{11}x_1 + \ldots + a_{1n}x_n \leq b_1 \\
&\quad \ldots \\
&\quad a_{m1}x_1 + \ldots + a_{mn}x_n \leq b_m \\
&\\
&\quad x_i \geq 0 \quad (1 \leq i \leq n)
\end{aligned}
$$

What is an Linear Program?

$$
\begin{aligned}
\min \quad & c^T x \\
s.t. \quad & Ax = b \\
& x \in \mathcal{R}^n_+
\end{aligned}
$$

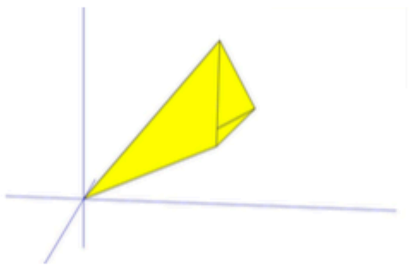
- How to generalize it?
  - moving to non-linear programs
  - keeping polynomial-time complexity

Conic Programming

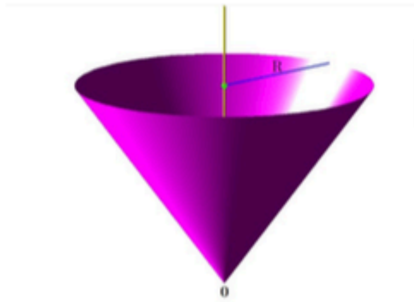
$$\begin{aligned} \min \quad & c^T x \\ s.t. \quad & Ax = b \\ & x \in \mathcal{K} \end{aligned}$$

- Key idea
  - replace the non-negativity constraints by a cone constraints
- Different types of cones
  - non-negativity constraints
  - second-order cones
  - semi-definite positive cone

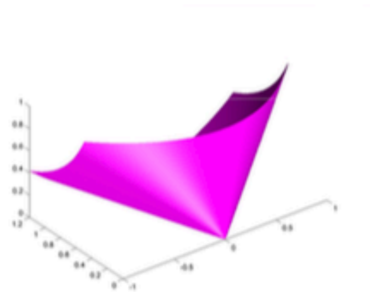
Proper Cones



(a) A polyhedron cone



(b) A Lorentz cone



(c) A positive semidefinite cone

Computational Complexity

- Old belief
  - the boundary between hard and easy problems depends on whether the problem is linear or not
- Finding of the last decades
  - convexity is the boundary between hard and easy problems in mathematical programming

Convexity

- A set $\mathcal{C}$ is convex

$$\forall x, y\ \mathcal{C}, \forall \lambda \in [0,1] : \lambda x + (1-\lambda) y \in \mathcal{C}$$

- A function $f : \Re^n \rightarrow \Re$ is convex if

$$\forall x, y \in \Re^n, \forall \lambda \in [0,1] : f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y)$$

- Formally, we should also make sure that dom f is convex

Convexity

- A function $f : \Re^n \rightarrow \Re$ is convex if

$$\forall x, y \in \Re^n, \forall \lambda \in [0,1] : f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y)$$

- A function is convex if its epigraph is convex

$$\mathbf{epi} f = \{(x,t) \in \Re^{n+1} : f(x) \leq t\}$$

What is A Cone?

- A set K is a cone if

$$\forall x \in K, \theta \geq 0 : \theta x \in K$$

conic combinations

- A cone K is convex if

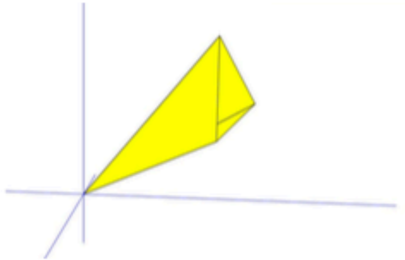
$$\forall x_1, x_2 \in K, \theta_1, \theta_2 \geq 0 : \theta_1 x_1 + \theta_2 x_2 \in K$$

- A cone K is pointed if it contains no line

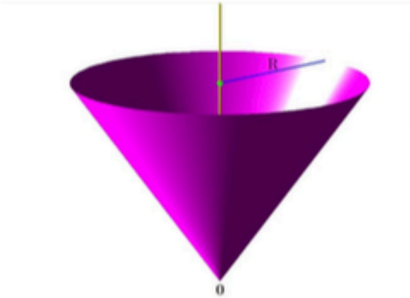
$$\forall x \in K : -x \in K \Rightarrow x = 0$$

- A cone is proper if it is closed, solid, pointed, and convex
  – closed = “contains its boundaries”, solid = “non empty interior”

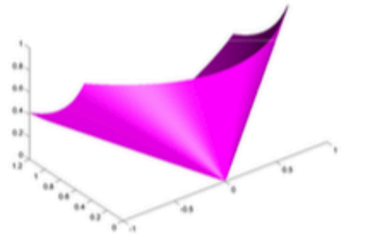
Proper Cones



(a) A polyhedron cone



(b) A Lorentz cone



(c) A positive semidefinite cone

Norms

- A function $f : \Re^n \to \Re$ is a norm on $\Re^n$ if
  – f is convex
  – f is positively homogeneous, i.e.,

$$\forall x \in \Re^n, \alpha \in \Re_+ : f(\alpha x) = \alpha f(x)$$

  – f is positive definite, i.e.,

$$\forall x \in \Re^n : f(x) = 0 \Rightarrow x = 0$$

- Consequences: A norm satisfies the triangle inequality (proof)

$$\forall x, y \in \Re^n : f(x + y) \leq f(x) + f(y)$$

Euclidian Norm

- Scalar or dot product

$$x^T y = \sum_{i=1}^{n} x_i y_i$$

- Euclidian norm ($x \in \Re^n$)

$$\|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2} = \sqrt{x^T x}$$

- Cauchy-Schwartz inequality

$$\forall x, y \in \Re^n : x^T y \leq \|x\|_2 \|y\|_2$$

The Euclidian Norm is convex

$$\|\alpha x + (1-\alpha)y\| \leqslant \alpha\|x\| + (1-\alpha)\|y\|$$

$$\sum_{i=1}^{n}(\alpha x_i + (1-\alpha)y_i)^2 \leqslant (\alpha\|x\| + (1-\alpha)\|y\|)^2$$

$$\alpha^2\|x\|^2 + (1-\alpha)^2\|y\|^2 + 2\alpha(1-\alpha)x^T y \leqslant \alpha^2\|x\|^2 + (1-\alpha)^2\|y\|^2 + 2\alpha(1-\alpha)\|x\|\|y\|$$

Ice Cream

Second-Order Cone

- Definition of the Ice-Cream Cone (aka Lorentz)

$$C_k = \left\{ \begin{bmatrix} u \\ t \end{bmatrix} \mid u \in \Re^{k-1}, t \in \Re, \|u\| \leq t \right\}$$

Convexity of the Second-Order Cone

- Definition of the Ice-Cream Cone (aka Lorentz)

$$C_k = \left\{ \begin{bmatrix} u \\ t \end{bmatrix} \mid u \in \Re^{k-1}, t \in \Re, \|u\| \leq t \right\}$$

- Why is this a convex set?

Second-Order Cone Programming

$$
\begin{array}{ll}
\min & c^T x \\
s.t. & Ax = b \\
& x \in \mathcal{C}_{n_1} \times \ldots \times \mathcal{C}_{n_k} \\
& (n_1 + \ldots + n_k = n)
\end{array}
$$

Second-Order Cone Programming

$$
\begin{array}{lll}
\min & f^T x & \\
s.t. & \|A_i x - b_i\| \leq c_i^T x_i - d_i & 1 \leq i \leq m
\end{array}
$$

Second-Order Cone Programming

$$
\begin{array}{lll}
\min & f^T x & \\
s.t. & \|A_i x - b_i\| \leq c_i^T x_i - d_i & 1 \leq i \leq m
\end{array}
$$

$$\Updownarrow$$

$$
\begin{array}{lll}
\min & f^T x & \\
s.t. & A_i x - b_i = y_i & 1 \leq i \leq m \\
 & c_i^T x_i - d_i = z_i & 1 \leq i \leq m \\
 & \|y_i\| \leq z_i & 1 \leq i \leq m
\end{array}
$$

Rotated Second-Order Cone

- Consider the contraint

$$w^2 \leq xy, \;\; x \geq 0, \;\; y \geq 0$$

- It is equivalent to

$$\left\| \begin{matrix} 2w \\ x - y \end{matrix} \right\| \leq x + y$$

Rotated Second-Order Cone

- Consider the contraint

$$w^2 \leq xy, \;\; x \geq 0, \;\; y \geq 0$$

- It is equivalent to

$$\left\| \begin{matrix} 2w \\ x - y \end{matrix} \right\| \leq x + y$$

$$\sqrt{4w^2 + (x-y)^2} \leqslant x + y$$

$$4w^2 + (x-y)^2 \leqslant (x+y)^2$$

$$4w^2 \leqslant 4(xy)$$

Rotated Second-Order Cone

- Consider the contraint

$$w^T w \leq xy, \; x \geq 0, \; y \geq 0$$

- It is equivalent to

$$\left\| \begin{matrix} 2w \\ x - y \end{matrix} \right\| \leq x + y$$

How to solve SOCP?

- Consider a convex problem

$$
\begin{aligned}
\min \quad & f_0(x) \\
s.t. \quad & Ax = b \\
& f_i(x) \leq 0 \qquad 1 \leq i \leq m
\end{aligned}
$$

- The problem
  - twice differentiable
  - is strictly feasible (there is a solution in the interior of the convex region)

How to solve SOCP?

- Rewrite

$$\begin{aligned} \min \quad & f_0(x) \\ s.t. \quad & Ax = b \\ & f_i(x) \leq 0 \qquad 1 \leq i \leq m \end{aligned}$$

- into

$$\begin{aligned} \min \quad & f_0(x) + \sum_{i=1}^{m} I^-(f_i(x)) \\ s.t. \quad & Ax = b \end{aligned}$$

- where

$$I^-(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ \infty & \text{otherwise} \end{cases}$$

How to solve SOCP?

- Rewrite

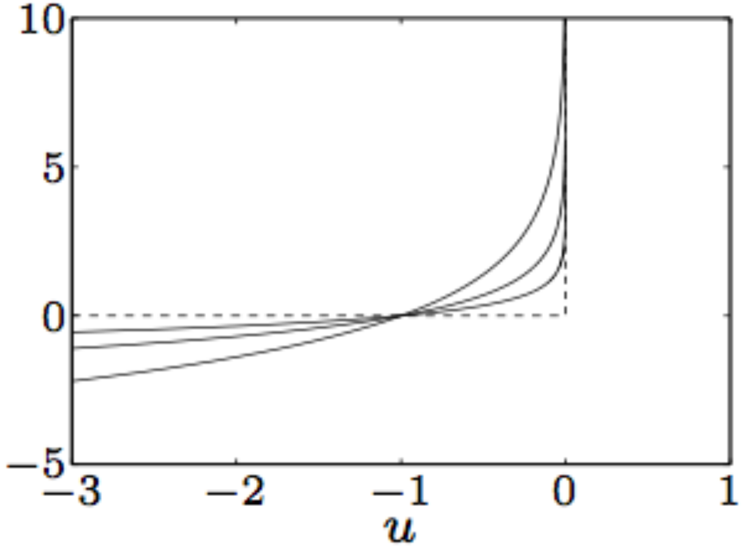
$$
\begin{aligned}
\min \quad & f_0(x) + \sum_{i=1}^{m} I^{-}(f_i(x)) \\
s.t. \quad & Ax = b
\end{aligned}
$$

- into

$$
\begin{aligned}
\min \quad & f_0(x) - \frac{1}{t} \sum_{i=1}^{m} \log(-f_i(x)) \\
s.t. \quad & Ax = b
\end{aligned}
$$

log barrier function

How to solve SOCP?



$$-\frac{1}{t}\log(-u)$$

Logarithmic Barrier Function

- Convex function

$$\phi(x) = -\sum_{i=1}^{m} \log(-f_i(x)) \qquad dom\ \phi = \{x \mid f_1(x) < 0, \ldots, f_m(x) < 0\}$$

- twice continuously differentiable

$$\nabla \phi(x) = -\sum_{i=1}^{m} \frac{1}{f_i(x)} \nabla f_i(x)$$

$$\nabla^2 \phi(x) = \sum_{i=1}^{m} \frac{1}{f_i(x)^2} \nabla f_i(x) \nabla f_i(x)^T - \sum_{i=1}^{m} \frac{1}{f_i(x)} \nabla^2 f_i(x)$$

How to solve SOCP?

- For SOCP, the optimization becomes

$$
\begin{array}{ll}
\min & c^T x - \sum_{i=1}^{m} \log(t^2 - \|x\|^2) \\
s.t. & Ax = b
\end{array}
$$

- Properties
  - It is finite if $\|x\| < t$
  - It converges to infinity as (x,t) approaches the boundary of the cone.
- Use a primal-dual method with this barrier function