DTU Summer School on Modern Optimization in Energy Systems

The QC Relaxation of OPF

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Physical Laws

- Kirchhoff's current law

$$I_i^g - \boldsymbol{I}_i^d = \sum_{(i,j) \in E \cup E^R} I_{ij}$$

- Ohm's law

$$I_{ij} = \boldsymbol{Y}_{ij}(V_i - V_j)$$

- AC Power

$$S_{ij} = V_i I_{ij}^*$$

AC Power Flows

$$S_i^g - \boldsymbol{S}_i^d = \sum_{(i,j) \in E \cup E^R} S_{ij} \quad \forall i \in N$$

$$S_{ij} = \boldsymbol{Y}_{ij}^* \boxed{V_i V_i^*} - \boldsymbol{Y}_{ij}^* \boxed{V_i V_j^*} \quad (i,j) \in E \cup E^R$$

W-Formulation

$$W_{ij} = V_i V_j^* \quad \forall i \in N, \forall j \in N$$

$$S_i^g - \boldsymbol{S}_i^d = \sum_{(i,j) \in E \cup E^R} S_{ij} \quad \forall i \in N$$

$$S_{ij} = \boldsymbol{Y}_{ij}^* W_{ii} - \boldsymbol{Y}_{ij}^* W_{ij} \quad (i,j) \in E$$

$$S_{ji} = \boldsymbol{Y}_{ij}^* W_{jj} - \boldsymbol{Y}_{ij}^* W_{ij}^* \quad (i,j) \in E$$

The SOCP Relaxation

- Jabr, R.: Radial distribution load flow using conic programming. (2006)

$$W_{ij} = V_i V_j^*$$

$$W_{ij} W_{ij}^* = V_i V_j^* V_i^* V_j$$

$$|W_{ij}|^2 = W_{ii} W_{jj}$$

$$\boxed{|W_{ij}|^2 \leqslant W_{ii} W_{jj}}$$

The SOCP Relaxation

$$S_i^g - \mathbf{S}_i^d = \sum_{(i,j)\in E\cup E^R} S_{ij} \qquad (i \in N)$$

$$S_{ij} = \mathbf{Y}_{ij}^* W_{ii} - \mathbf{Y}_{ij}^* W_{ij} \qquad (i,j) \in E$$

$$S_{ji} = \mathbf{Y}_{ij}^* W_{jj} - \mathbf{Y}_{ij}^* W_{ij}^* \qquad (i,j) \in E$$

$$|W_{ij}|^2 \leq W_{ii} W_{jj}$$

The SOCP Relaxation

- Beautiful work

R. Jabr, "Radial distribution load flow using conic programming," *IEEE Transactions on Power Systems*, vol. 21, no. 3, pp. 1458–1459, Aug 2006.

- Polar form

$$p_{ij} = \boldsymbol{g}_i v_i^2 - \boldsymbol{g}_i v_i v_j \cos(\theta_{ij}) - \boldsymbol{b}_i v_i v_j \sin(\theta_{ij})$$
$$q_{ij} = -\boldsymbol{b}_i v_i^2 + \boldsymbol{b}_i v_i v_j \cos(\theta_{ij}) - \boldsymbol{g}_i v_i v_j \sin(\theta_{ij})$$

- Define

$$w_{ij}^R = v_i v_j \cos\theta_{ij}$$
$$w_{ij}^I = v_i v_j \sin\theta_{ij}$$

The SOCP Relaxation

- The power equations become

$$p_{ij} = \mathbf{g}_{ij} v_i^2 - \mathbf{g}_{ij} w_{ij}^R - \mathbf{b}_{ij} w_{ij}^I$$
$$q_{ij} = -\mathbf{b}_{ij} v_i^2 + \mathbf{b}_{ij} w_{ij}^R - \mathbf{g}_{ij} w_{ij}^I$$
$$(w_{ij}^R)^2 + (w_{ij}^I)^2 \leq v_i^2 v_j^2$$

Power Losses

Power Losses

$$
\begin{aligned}
S_{ij} + S_{ji} =& V_i I_{ij}^* + V_j I_{ji}^* \\
& V_i \mathbf{Y}_{ij}^* (V_i - V_j)^* + V_j \mathbf{Y}_{ij}^* (V_j - V_i)^* \\
& V_i \mathbf{Y}_{ij}^* (V_i - V_j)^* - V_j \mathbf{Y}_{ij}^* (V_i - V_j)^* \\
& \mathbf{Y}_{ij}^* (V_i - V_j)^* (V_i - V_j)
\end{aligned}
$$

Power Losses Again

$$
\begin{aligned}
S_{ij} + S_{ji} &= \mathbf{Y}_{ij}^*(V_i - V_j)^*(V_i - V_j) \\
&= \mathbf{Z}_{ij}\mathbf{Y}_{ij}\mathbf{Y}_{ij}^*(V_i - V_j)^*(V_i - V_j) \\
&= \mathbf{Z}_{ij}\mathbf{Y}_{ij}(V_i - V_j)\mathbf{Y}_{ij}^*(V_i - V_j)^* \\
&= \mathbf{Z}_{ij}I_{ij}I_{ij}^* \\
&= \mathbf{Z}_{ij}I_{ij}^2 \\
&= \mathbf{Z}_{ij}l_{ij}
\end{aligned}
$$

Another SOCP Relaxation

- Power Losses

$$S_{ij} + S_{ji} \quad = \mathbf{Z}_{ij} l_{ij}$$

- Second-Order Constraint

$$S_{ij} = V_i I_{ij}^*$$

$$S_{ij} S_{ij}^* = V_i I_{ij}^* V_i^* I_{ij}$$

$$|S_{ij}|^2 = W_{ii} l_{ij}$$

$$|S_{ij}|^2 \leq W_{ii} l_{ij}$$

Another SOCP Relaxation

$$S_i^g - \mathbf{S}_i^d = \sum_{(i,j)\in E\cup E^R} S_{ij} \qquad (i \in N)$$

$$S_{ij} = \mathbf{Y}_{ij}^* W_{ii} - \mathbf{Y}_{ij}^* W_{ij} \qquad (i,j) \in E$$

$$S_{ji} = \mathbf{Y}_{ij}^* W_{jj} - \mathbf{Y}_{ij}^* W_{ij}^* \qquad (i,j) \in E$$

$$|S_{ij}|^2 \leq W_{ii} l_{ij}$$

$$S_{ij} + S_{ji} = \mathbf{Z}_{ij} l_{ij}$$

SOCP Relaxations

- The two SOCP relaxations are equivalent
  - Farivar, M., Clarke, C., Low, S., Chandy, K. 2011