DTU Summer School on Modern Optimization in Energy Systems

Conic Optimization: Part III

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Goals of the Lecture

- Conic Optimization
  – Semi-Definite Programming over Complex Numbers

AC Power Flows

$$S_i^g - \boldsymbol{S}_i^d = \sum_{(i,j)\in E \cup E^R} S_{ij} \quad \forall i \in N$$

$$S_{ij} = \boldsymbol{Y}_{ij}^* \boxed{V_i V_i^*} - \boldsymbol{Y}_{ij}^* \boxed{V_i V_j^*} \quad (i,j) \in E \cup E^R$$

Motivation

- Can we generalize SDP over complex matrices?
  - what is the equivalent of a symmetric matrix?
  - what is the equivalent of the SDP constraint?

Hermitian Matrices

1855

Hermitian Matrices

Hermitian Matrices

- The Adjoint/Conjugate M* of a complex matrix is defined as

$$[m_{ij}^*]^T$$

- A complex matrix is Hermitian if

$$M = M^*$$

- We use H_n to denote the set of n by n Hermitian matrices

- Inner product over $\mathbb{C}^n$

$$\langle u, v \rangle = v^* u$$

Hermitian Matrices

- Traces of Hermitian Matrices

$$\langle A, B \rangle = Tr(B^* A) = \sum_{i,j} a_{ij} b^*_{ij}$$

- Observe that

$$\langle A, B \rangle = Tr(B^* A) = Tr(BA) = Tr(AB)$$

- Property

$$(AB)^* = B^* A^*$$

Hermitian Matrices

- Hermitian Matrices have only real eigenvalues

$$Ax = \lambda x$$

$$\boxed{x^* A x = \lambda x^* x}$$

$$(x^* A x)^* = (\lambda x^* x)^*$$

$$(x^* A x)^* = \boxed{x^* A^* x = \lambda^* x^* x}$$

$$(\lambda - \lambda^*)(x^* x) = 0 \Rightarrow \lambda = \lambda^*$$

Positive Semi-Definite Matrices

- An n by n Hermitian matrix M is positive semi-definitive if
  - $x^* M x \geq 0 \text{ for all } x \in \mathbb{C}^n$
  - all its eigenvalues are positive
  - M = V V* for some V in H_n
- The last condition can be rephrased as
  - M is positive semi-definite if and only if there exist vectors

$$v_1, \ldots, v_n \in \mathbb{C}^n$$

  - such that

$$m_{ij} = \langle v_i, v_j \rangle$$

Hermitian Matrix

Complex Semi-Definite Programming

- Canonical Form

$$\begin{aligned} \min_{X \in H_n} \quad & Tr(CX) \\ s.t. \quad & Tr(A_i X) = b_i \qquad 1 \leq i \leq m \\ & X \succeq 0 \end{aligned}$$

Complex SDP

Complex SDP can be written into Real SDP

From Complex SDP to Real SDP

- Let X in H_n. Define the matrix transformation

$$\mathcal{L}(X) = \begin{bmatrix} \Re(X) & -\Im(X) \\ \Im(X) & \Re(X) \end{bmatrix}$$

- where

$$\Re(X) = [\Re(x_{ij})]$$

$$\Im(X) = [\Im(x_{ij})]$$

From Complex SDP to Real SDP

- Let X in H_n. Define the matrix transformation

$$\mathcal{L}(X) = \begin{bmatrix} \Re(X) & -\Im(X) \\ \Im(X) & \Re(X) \end{bmatrix}$$

is symmetric

- where

$$\Re(X) = [\Re(x_{ij})]$$

is symmetric

$$\Im(X) = [\Im(x_{ij})]$$

is skew-symmetric $(X = -X^T)$

From Complex SDP to Real SDP

- Let A, B in H_n.

$$\langle \mathcal{L}(A), \mathcal{L}(B) \rangle = Tr(\begin{bmatrix} \Re(B) & \Im(B) \\ -\Im(B) & \Re(B) \end{bmatrix} \begin{bmatrix} \Re(A) & \Im(A) \\ -\Im(A) & \Re(A) \end{bmatrix})$$

$$\langle \mathcal{L}(A), \mathcal{L}(B) \rangle = 2Tr(\Re(A)\Re(B) + \Im(A)\Im(B)) = 2\langle A, B \rangle$$

From Complex SDP to Real SDP

- Key idea
  - replace X by $\mathcal{L}(\mathcal{X})$
- This requires
  - applying similar transformation to all constraints
  - adding constraints on the shape of the new matrix

From Complex SDP to Real SDP

$$
\begin{aligned}
\min_{Y \in S_{2n}} \quad & Tr(\mathcal{L}(C)Y) & \\
s.t. \quad & Tr(\mathcal{L}(A_i)Y) \leq 2b_i & 1 \leq i \leq m \\
& Y \succeq 0 & \\
& \langle \begin{bmatrix} E_{ij} & 0 \\ 0 & -E_{ij} \end{bmatrix}, Y \rangle = 0 & 1 \leq i,j \leq n \\
& \langle \begin{bmatrix} 0 & E_{ij} \\ E_{ij} & 0 \end{bmatrix}, Y \rangle = 0 & 1 \leq i,j \leq n
\end{aligned}
$$

- E_{ij} is a matrix that has a 1 in position (i,j) and (j,i) and is zero otherwise

From Complex SDP to Real SDP

- What are the last two constraints ensuring?

$$Y = \begin{bmatrix} A & -B \\ B & A \end{bmatrix}$$

- Note that if I have Y, I can recover X.
  – How?

From Complex SDP to Real SDP

- If X is feasible, then Y=L(X) is feasible
- If Y is feasible, then $X=L^{-1}(Y)$ is feasible
- The objective value for the real problem is twice the objective value of the complex subproblem.

From Complex SDP to Real SDP

- A complex eigenvector x for the complex subproblem is associated with the real eigenvector y

$$y = \begin{pmatrix} \Re(x) \\ \Im(x) \end{pmatrix}$$

- A real eigenvector $(r \ m)^T$ is associated with a complex vector

$$x = r + im$$