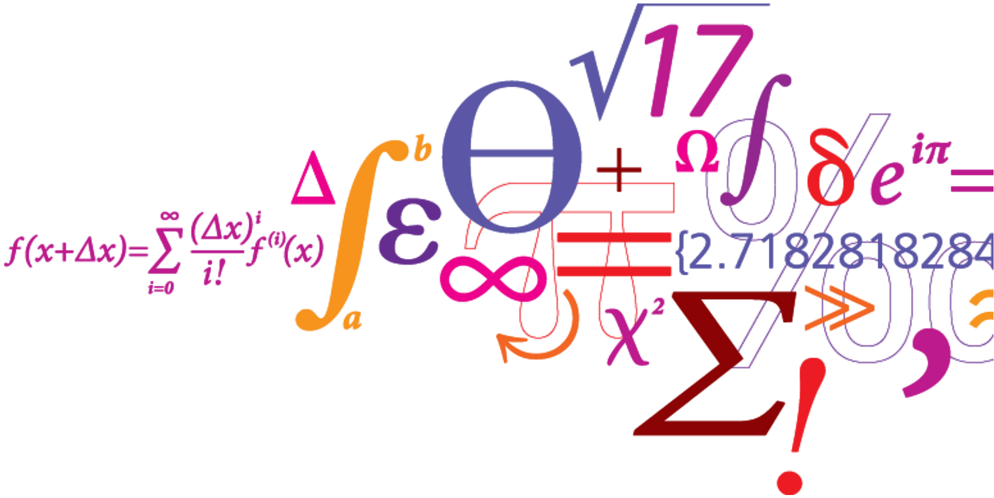
Market Design for Integrated Energy Systems

Part II: Optimization, Equilibrium, Properties

Jalal Kazempour



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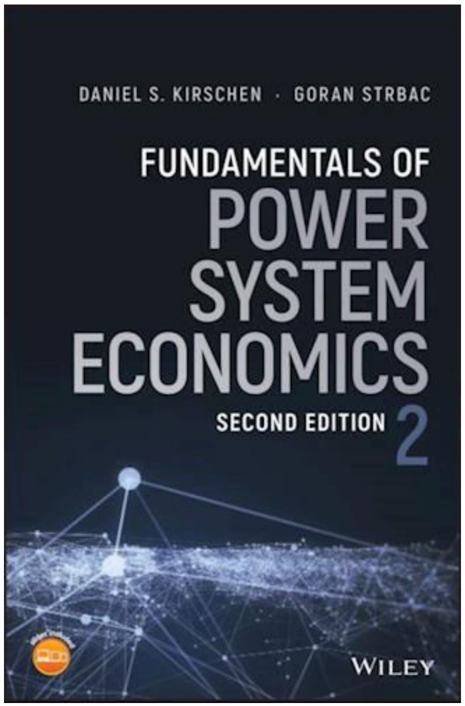
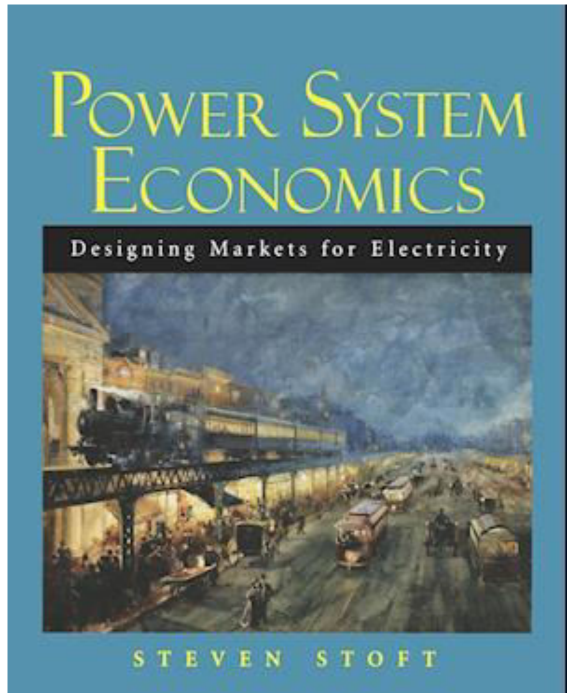
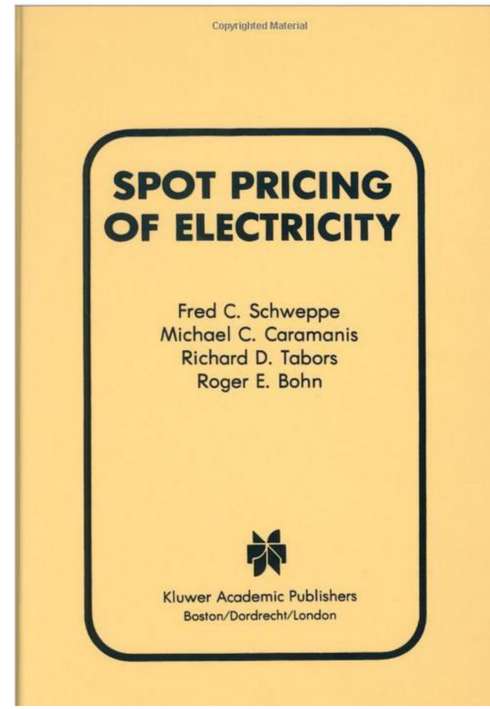
DTU Summer School 2022

Very nice books about electricity markets







Market clearing: a simple example

An extended example: two generators (G1 and G2) and two elastic demands (D1 and D2)

Capacity: 100 MW
Offer price: $12/MWh

Capacity: 80 MW
Offer price: $20/MWh

Demand D1

Maximum load: 100 MW
Bid price: $40/MWh

Demand D2

Maximum load: 50 MW
Bid price: $35/MWh

Market outcomes:

- Productions of G1 and G2: ?
- Consumptions of D1 and D2: ?
- Market-clearing price: ?

Market clearing: a simple example

An extended example: two generators (G1 and G2) and two elastic demands (D1 and D2)

Capacity: 100 MW
Offer price: $12/MWh

Capacity: 80 MW
Offer price: $20/MWh

Demand D1

Maximum load: 100 MW
Bid price: $40/MWh

Demand D2

Maximum load: 50 MW
Bid price: $35/MWh

Market outcomes:

- Productions of G1 and G2: **100 MW and 50 MW**
- Consumptions of D1 and D2: **100 MW and 50 MW**
- Market-clearing price: **[20-35] $/MWh**

Market clearing as an optimization problem

$$\underset{p^{\mathrm{G1}},p^{\mathrm{G2}},p^{\mathrm{D1}},p^{\mathrm{D2}}}{\text{Maximize}} \quad SW = [40p^{\mathrm{D1}} + 35p^{\mathrm{D2}}] - [12p^{\mathrm{G1}} + 20p^{\mathrm{G2}}] \tag{1a}$$

subject to:

$$0 \leq p^{\mathrm{D1}} \leq 100 \tag{1b}$$

$$0 \leq p^{\mathrm{D2}} \leq 50 \tag{1c}$$

$$0 \leq p^{\mathrm{G1}} \leq 100 \tag{1d}$$

$$0 \leq p^{\mathrm{G2}} \leq 80 \tag{1e}$$

$$p^{\mathrm{D1}} + p^{\mathrm{D2}} - p^{\mathrm{G1}} - p^{\mathrm{G2}} = 0 \qquad : \lambda \tag{1f}$$

Market clearing as an optimization problem

$$\underset{p^{G1},p^{G2},p^{D1},p^{D2}}{\text{Maximize}} \quad SW = \underbrace{[40p^{D1} + 35p^{D2}]}_{\text{Utility of demands}} - \underbrace{[12p^{G1} + 20p^{G2}]}_{\text{Cost of generators}} \tag{1a}$$

subject to:

$$0 \leq p^{D1} \leq 100 \tag{1b}$$

$$0 \leq p^{D2} \leq 50 \tag{1c}$$

Consumption limits

$$0 \leq p^{G1} \leq 100 \tag{1d}$$

$$0 \leq p^{G2} \leq 80 \tag{1e}$$

Generation limits

$$\underbrace{p^{D1} + p^{D2} - p^{G1} - p^{G2} = 0}_{\text{Power balance equality}} \quad : \lambda \tag{1f}$$

Dual variable: market-clearing price

Discussion

Question:

How to make sure all market participants (i.e., G1, G2, D1 and D2) are satisfied with the market-clearing outcome, and would not deviate from it?

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Let's develop an optimization problem for each market participant!

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Question:

What is the objective of each generator?

What is the objective of each elastic demand?

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Let's develop an optimization problem for each market participant!

Question:

What is the objective of each generator? Profit maximization!

What is the objective of each elastic demand? Utility maximization!

Discussion

Question:

How to make sure all market participants (i.e., G1, G2, D1 and D2) are satisfied with the market-clearing outcome, and would not deviate from it?

Let's develop an optimization problem for each market participant!

Question:

What is the objective of each generator? Profit maximization!

What is the objective of each elastic demand? Utility maximization!

Question:

How to calculate a generator's profit?

How to calculate an elastic demand's utility?

Discussion

Question:

How to make sure all market participants (i.e., G1, G2, D1 and D2) are satisfied with the market-clearing outcome, and would not deviate from it?

Let's develop an optimization problem for each market participant!

Question:

What is the objective of each generator? Profit maximization!

What is the objective of each elastic demand? Utility maximization!

Question:

How to calculate a generator's profit? Production level x [market price – production cost]

How to calculate an elastic demand's utility? Consumption level x [bid price – market price]

Optimization problem for each market player

Optimization problem for each market player

For generator G1:

$$\underset{p^{\mathrm{G1}}}{\text{Maximize}} \quad p^{\mathrm{G1}}(\lambda - 12)$$

subject to:

$$0 \leq p^{\mathrm{G1}} \leq 100 \quad : \underline{\mu}^{\mathrm{G1}}, \overline{\mu}^{\mathrm{G1}}$$

Optimization problem for each market player

For generator G1:

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For generator G2:

$$\underset{p^{\mathrm{G2}}}{\text{Maximize}} \quad p^{\mathrm{G2}}(\lambda - 20)$$

subject to:

$$0 \leq p^{\mathrm{G2}} \leq 80 \quad : \underline{\mu}^{\mathrm{G2}}, \overline{\mu}^{\mathrm{G2}}$$

Optimization problem for each market player

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$$\underset{p^{\mathrm{G1}}}{\text{Maximize}} \quad p^{\mathrm{G1}}(\lambda - 12)$$

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subject to:

$$0 \leq p^{\mathrm{G2}} \leq 80 \quad : \underline{\mu}^{\mathrm{G2}}, \overline{\mu}^{\mathrm{G2}}$$

For demand D1:

$$\underset{p^{\mathrm{D1}}}{\text{Maximize}} \quad p^{\mathrm{D1}}(40 - \lambda)$$

subject to:

$$0 \leq p^{\mathrm{D1}} \leq 100 \quad : \underline{\mu}^{\mathrm{D1}}, \overline{\mu}^{\mathrm{D1}}$$

Optimization problem for each market player

For generator G1:

$$\underset{p^{G1}}{\text{Maximize}} \quad p^{G1}(\lambda - 12)$$

subject to:

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For generator G2:

$$\underset{p^{G2}}{\text{Maximize}} \quad p^{G2}(\lambda - 20)$$

subject to:

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subject to:

$$0 \le p^{D1} \le 100 \quad : \underline{\mu}^{D1}, \overline{\mu}^{D1}$$

For demand D2:

$$\underset{p^{D2}}{\text{Maximize}} \quad p^{D2}(35 - \lambda)$$

subject to:

$$0 \le p^{D2} \le 50 \quad : \underline{\mu}^{D2}, \overline{\mu}^{D2}$$

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For generator G1:

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Market price λ is a given value (treated as a parameter) within each optimization problem!

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subject to:

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Market price λ is a given value (treated as a parameter) within each optimization problem!

Question:
How do market players contribute to market price formation? Do we need an extra condition?

Optimization problem for each market player

For generator G1:

$$\underset{p^{G1}}{\text{Maximize}} \quad p^{G1}(\lambda - 12)$$

subject to:

$$0 \leq p^{G1} \leq 100 \quad : \underline{\mu}^{G1}, \overline{\mu}^{G1}$$

For generator G2:

$$\underset{p^{G2}}{\text{Maximize}} \quad p^{G2}(\lambda - 20)$$

subject to:

$$0 \leq p^{G2} \leq 80 \quad : \underline{\mu}^{G2}, \overline{\mu}^{G2}$$

For demand D1:

$$\underset{p^{D1}}{\text{Maximize}} \quad p^{D1}(40 - \lambda)$$

subject to:

$$0 \leq p^{D1} \leq 100 \quad : \underline{\mu}^{D1}, \overline{\mu}^{D1}$$

For demand D2:

$$\underset{p^{D2}}{\text{Maximize}} \quad p^{D2}(35 - \lambda)$$

subject to:

$$0 \leq p^{D2} \leq 50 \quad : \underline{\mu}^{D2}, \overline{\mu}^{D2}$$

Market price λ is a given value (treated as a parameter) within each optimization problem!

Question:
How do market players contribute to market price formation? Do we need an extra condition? Yes!

Optimization problem for each market player

For generator G1:

$$\underset{p^{\mathrm{G1}}}{\text{Maximize}} \quad p^{\mathrm{G1}}(\lambda - 12)$$

subject to:

$$0 \leq p^{\mathrm{G1}} \leq 100 \quad : \underline{\mu}^{\mathrm{G1}}, \overline{\mu}^{\mathrm{G1}}$$

For generator G2:

$$\underset{p^{\mathrm{G2}}}{\text{Maximize}} \quad p^{\mathrm{G2}}(\lambda - 20)$$

subject to:

$$0 \leq p^{\mathrm{G2}} \leq 80 \quad : \underline{\mu}^{\mathrm{G2}}, \overline{\mu}^{\mathrm{G2}}$$

For demand D1:

$$\underset{p^{\mathrm{D1}}}{\text{Maximize}} \quad p^{\mathrm{D1}}(40 - \lambda)$$

subject to:

$$0 \leq p^{\mathrm{D1}} \leq 100 \quad : \underline{\mu}^{\mathrm{D1}}, \overline{\mu}^{\mathrm{D1}}$$

For demand D2:

$$\underset{p^{\mathrm{D2}}}{\text{Maximize}} \quad p^{\mathrm{D2}}(35 - \lambda)$$

subject to:

$$0 \leq p^{\mathrm{D2}} \leq 50 \quad : \underline{\mu}^{\mathrm{D2}}, \overline{\mu}^{\mathrm{D2}}$$

Power balance equality:

$$p^{\mathrm{D1}} + p^{\mathrm{D2}} - p^{\mathrm{G1}} - p^{\mathrm{G2}} = 0 \quad : \lambda$$

Optimization problem for each market player

For generator G1:

$$\underset{p^{G1}}{\text{Maximize}} \quad p^{G1}(\lambda - 12)$$

subject to:

$$0 \leq p^{G1} \leq 100 \quad : \underline{\mu}^{G1}, \overline{\mu}^{G1}$$

For generator G2:

$$\underset{p^{G2}}{\text{Maximize}} \quad p^{G2}(\lambda - 20)$$

subject to:

$$0 \leq p^{G2} \leq 80 \quad : \underline{\mu}^{G2}, \overline{\mu}^{G2}$$

For demand D1:

$$\underset{p^{D1}}{\text{Maximize}} \quad p^{D1}(40 - \lambda)$$

subject to:

$$0 \leq p^{D1} \leq 100 \quad : \underline{\mu}^{D1}, \overline{\mu}^{D1}$$

For demand D2:

$$\underset{p^{D2}}{\text{Maximize}} \quad p^{D2}(35 - \lambda)$$

subject to:

$$0 \leq p^{D2} \leq 50 \quad : \underline{\mu}^{D2}, \overline{\mu}^{D2}$$

Power balance equality:

$$p^{D1} + p^{D2} - p^{G1} - p^{G2} = 0 \quad : \lambda$$

Question: From mathematical perspective, is the above power balance equality "equivalent" to the following "unconstrained" optimization problem? Why?

$$\underset{\lambda}{\text{Minimize}} \quad \lambda \left(p^{D1} + p^{D2} - p^{G1} - p^{G2}\right)$$

Optimization problem for each market player

For generator G1:

$$\underset{p^{\mathrm{G1}}}{\text{Maximize}} \quad p^{\mathrm{G1}}(\lambda - 12)$$

subject to:

$$0 \leq p^{\mathrm{G1}} \leq 100 \quad : \underline{\mu}^{\mathrm{G1}}, \overline{\mu}^{\mathrm{G1}}$$

For generator G2:

$$\underset{p^{\mathrm{G2}}}{\text{Maximize}} \quad p^{\mathrm{G2}}(\lambda - 20)$$

subject to:

$$0 \leq p^{\mathrm{G2}} \leq 80 \quad : \underline{\mu}^{\mathrm{G2}}, \overline{\mu}^{\mathrm{G2}}$$

For demand D1:

$$\underset{p^{\mathrm{D1}}}{\text{Maximize}} \quad p^{\mathrm{D1}}(40 - \lambda)$$

subject to:

$$0 \leq p^{\mathrm{D1}} \leq 100 \quad : \underline{\mu}^{\mathrm{D1}}, \overline{\mu}^{\mathrm{D1}}$$

For demand D2:

$$\underset{p^{\mathrm{D2}}}{\text{Maximize}} \quad p^{\mathrm{D2}}(35 - \lambda)$$

subject to:

$$0 \leq p^{\mathrm{D2}} \leq 50 \quad : \underline{\mu}^{\mathrm{D2}}, \overline{\mu}^{\mathrm{D2}}$$

Power balance equality:

$$p^{\mathrm{D1}} + p^{\mathrm{D2}} - p^{\mathrm{G1}} - p^{\mathrm{G2}} = 0 \quad : \lambda$$

Assume this unconstrained optimization problem is being solved by a fictitious player, the so-called "**price-setter**", who determines the market-clearing price by penalizing the power mismatch!

$$\underset{\lambda}{\text{Minimize}} \quad \lambda \left(p^{\mathrm{D1}} + p^{\mathrm{D2}} - p^{\mathrm{G1}} - p^{\mathrm{G2}}\right)$$

Optimization problem for each market player

For generator G1:

$$\underset{p^{G1}}{\text{Maximize}} \quad p^{G1}(\lambda - 12)$$

subject to:

$$0 \leq p^{G1} \leq 100 \quad : \underline{\mu}^{G1}, \overline{\mu}^{G1}$$

For generator G2:

$$\underset{p^{G2}}{\text{Maximize}} \quad p^{G2}(\lambda - 20)$$

subject to:

$$0 \leq p^{G2} \leq 80 \quad : \underline{\mu}^{G2}, \overline{\mu}^{G2}$$

For demand D1:

$$\underset{p^{D1}}{\text{Maximize}} \quad p^{D1}(40 - \lambda)$$

subject to:

$$0 \leq p^{D1} \leq 100 \quad : \underline{\mu}^{D1}, \overline{\mu}^{D1}$$

For demand D2:

$$\underset{p^{D2}}{\text{Maximize}} \quad p^{D2}(35 - \lambda)$$

subject to:

$$0 \leq p^{D2} \leq 50 \quad : \underline{\mu}^{D2}, \overline{\mu}^{D2}$$

Price-setter:

$$\underset{\lambda}{\text{Minimize}} \quad \lambda\left(p^{D1} + p^{D2} - p^{G1} - p^{G2}\right)$$

Optimization problem for each market player

For generator G1:

$$\underset{p^{G1}}{\text{Maximize}} \quad p^{G1}(\lambda - 12)$$

subject to:

$$0 \leq p^{G1} \leq 100 \quad : \underline{\mu}^{G1}, \overline{\mu}^{G1}$$

For generator G2:

$$\underset{p^{G2}}{\text{Maximize}} \quad p^{G2}(\lambda - 20)$$

subject to:

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For demand D1:

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For demand D2:

$$\underset{p^{D2}}{\text{Maximize}} \quad p^{D2}(35 - \lambda)$$

subject to:

$$0 \leq p^{D2} \leq 50 \quad : \underline{\mu}^{D2}, \overline{\mu}^{D2}$$

Price-setter:

$$\underset{\lambda}{\text{Minimize}} \quad \lambda \left(p^{D1} + p^{D2} - p^{G1} - p^{G2}\right)$$

Question: Can we solve optimization problems above separately?

Optimization problem for each market player

For generator G1:

$$\underset{p^{G1}}{\text{Maximize}} \quad p^{G1}(\lambda - 12)$$
subject to:
$$0 \leq p^{G1} \leq 100 \quad : \underline{\mu}^{G1}, \overline{\mu}^{G1}$$

For generator G2:

$$\underset{p^{G2}}{\text{Maximize}} \quad p^{G2}(\lambda - 20)$$
subject to:
$$0 \leq p^{G2} \leq 80 \quad : \underline{\mu}^{G2}, \overline{\mu}^{G2}$$

For demand D1:

$$\underset{p^{D1}}{\text{Maximize}} \quad p^{D1}(40 - \lambda)$$
subject to:
$$0 \leq p^{D1} \leq 100 \quad : \underline{\mu}^{D1}, \overline{\mu}^{D1}$$

For demand D2:

$$\underset{p^{D2}}{\text{Maximize}} \quad p^{D2}(35 - \lambda)$$
subject to:
$$0 \leq p^{D2} \leq 50 \quad : \underline{\mu}^{D2}, \overline{\mu}^{D2}$$

Price-setter:

$$\underset{\lambda}{\text{Minimize}} \quad \lambda \left(p^{D1} + p^{D2} - p^{G1} - p^{G2}\right)$$

Question: Can we solve optimization problems above separately? No! Why?

Optimization problem for each market player

For generator G1:

$$\underset{p^{G1}}{\text{Maximize}} \quad p^{G1}(\lambda - 12)$$

subject to:

$$0 \leq p^{G1} \leq 100 \quad : \underline{\mu}^{G1}, \overline{\mu}^{G1}$$

For generator G2:

$$\underset{p^{G2}}{\text{Maximize}} \quad p^{G2}(\lambda - 20)$$

subject to:

$$0 \leq p^{G2} \leq 80 \quad : \underline{\mu}^{G2}, \overline{\mu}^{G2}$$

For demand D1:

$$\underset{p^{D1}}{\text{Maximize}} \quad p^{D1}(40 - \lambda)$$

subject to:

$$0 \leq p^{D1} \leq 100 \quad : \underline{\mu}^{D1}, \overline{\mu}^{D1}$$

For demand D2:

$$\underset{p^{D2}}{\text{Maximize}} \quad p^{D2}(35 - \lambda)$$

subject to:

$$0 \leq p^{D2} \leq 50 \quad : \underline{\mu}^{D2}, \overline{\mu}^{D2}$$

Price-setter:

$$\underset{\lambda}{\text{Minimize}} \quad \lambda \left(p^{D1} + p^{D2} - p^{G1} - p^{G2}\right)$$

- Market-clearing price is a variable for the price-setter, but a parameter for G1, G2, D1 and D2.
- Productions/consumptions are variables for G1, G2, D1 and D2, but parameters for the price-setter.

Optimization problem for each market player

For generator G1:

$$\underset{p^{G1}}{\text{Maximize}} \quad p^{G1}(\lambda - 12)$$

subject to:

$$0 \leq p^{G1} \leq 100 \quad : \underline{\mu}^{G1}, \overline{\mu}^{G1}$$

For generator G2:

$$\underset{p^{G2}}{\text{Maximize}} \quad p^{G2}(\lambda - 20)$$

subject to:

$$0 \leq p^{G2} \leq 80 \quad : \underline{\mu}^{G2}, \overline{\mu}^{G2}$$

For demand D1:

$$\underset{p^{D1}}{\text{Maximize}} \quad p^{D1}(40 - \lambda)$$

subject to:

$$0 \leq p^{D1} \leq 100 \quad : \underline{\mu}^{D1}, \overline{\mu}^{D1}$$

For demand D2:

$$\underset{p^{D2}}{\text{Maximize}} \quad p^{D2}(35 - \lambda)$$

subject to:

$$0 \leq p^{D2} \leq 50 \quad : \underline{\mu}^{D2}, \overline{\mu}^{D2}$$

Price-setter:

$$\underset{\lambda}{\text{Minimize}} \quad \lambda \left(p^{D1} + p^{D2} - p^{G1} - p^{G2}\right)$$

All five optimization problems above are linked, and should be solved all together!

Optimization problem for each market player

For generator G1:

$$\underset{p^{\mathrm{G1}}}{\text{Maximize}} \quad p^{\mathrm{G1}}(\lambda - 12)$$

subject to:

$$0 \leq p^{\mathrm{G1}} \leq 100 \quad : \underline{\mu}^{\mathrm{G1}}, \overline{\mu}^{\mathrm{G1}}$$

For generator G2:

$$\underset{p^{\mathrm{G2}}}{\text{Maximize}} \quad p^{\mathrm{G2}}(\lambda - 20)$$

subject to:

$$0 \leq p^{\mathrm{G2}} \leq 80 \quad : \underline{\mu}^{\mathrm{G2}}, \overline{\mu}^{\mathrm{G2}}$$

For demand D1:

$$\underset{p^{\mathrm{D1}}}{\text{Maximize}} \quad p^{\mathrm{D1}}(40 - \lambda)$$

subject to:

$$0 \leq p^{\mathrm{D1}} \leq 100 \quad : \underline{\mu}^{\mathrm{D1}}, \overline{\mu}^{\mathrm{D1}}$$

For demand D2:

$$\underset{p^{\mathrm{D2}}}{\text{Maximize}} \quad p^{\mathrm{D2}}(35 - \lambda)$$

subject to:

$$0 \leq p^{\mathrm{D2}} \leq 50 \quad : \underline{\mu}^{\mathrm{D2}}, \overline{\mu}^{\mathrm{D2}}$$

Price-setter:

$$\underset{\lambda}{\text{Minimize}} \quad \lambda \left(p^{\mathrm{D1}} + p^{\mathrm{D2}} - p^{\mathrm{G1}} - p^{\mathrm{G2}}\right)$$

This is a game-theoretic problem.

Optimization problem for each market player

For generator G1:

$$\begin{aligned}&\underset{p^{\mathrm{G1}}}{\text{Maximize}} \quad p^{\mathrm{G1}}(\lambda - 12)\\&\text{subject to:}\\&0 \leq p^{\mathrm{G1}} \leq 100 \quad : \underline{\mu}^{\mathrm{G1}}, \overline{\mu}^{\mathrm{G1}}\end{aligned}$$

For generator G2:

$$\begin{aligned}&\underset{p^{\mathrm{G2}}}{\text{Maximize}} \quad p^{\mathrm{G2}}(\lambda - 20)\\&\text{subject to:}\\&0 \leq p^{\mathrm{G2}} \leq 80 \quad : \underline{\mu}^{\mathrm{G2}}, \overline{\mu}^{\mathrm{G2}}\end{aligned}$$

For demand D1:

$$\begin{aligned}&\underset{p^{\mathrm{D1}}}{\text{Maximize}} \quad p^{\mathrm{D1}}(40 - \lambda)\\&\text{subject to:}\\&0 \leq p^{\mathrm{D1}} \leq 100 \quad : \underline{\mu}^{\mathrm{D1}}, \overline{\mu}^{\mathrm{D1}}\end{aligned}$$

For demand D2:

$$\begin{aligned}&\underset{p^{\mathrm{D2}}}{\text{Maximize}} \quad p^{\mathrm{D2}}(35 - \lambda)\\&\text{subject to:}\\&0 \leq p^{\mathrm{D2}} \leq 50 \quad : \underline{\mu}^{\mathrm{D2}}, \overline{\mu}^{\mathrm{D2}}\end{aligned}$$

Price-setter:

$$\underset{\lambda}{\text{Minimize}} \quad \lambda \left(p^{\mathrm{D1}} + p^{\mathrm{D2}} - p^{\mathrm{G1}} - p^{\mathrm{G2}}\right)$$

This is a game-theoretic problem.

This specific problem is also known as "competitive equilibrium" problem!

Optimization problem for each market player

For generator G1:

$$\underset{p^{G1}}{\text{Maximize}} \quad p^{G1}(\lambda - 12)$$

subject to:

$$0 \leq p^{G1} \leq 100 \quad : \underline{\mu}^{G1}, \overline{\mu}^{G1}$$

For generator G2:

$$\underset{p^{G2}}{\text{Maximize}} \quad p^{G2}(\lambda - 20)$$

subject to:

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For demand D1:

$$\underset{p^{D1}}{\text{Maximize}} \quad p^{D1}(40 - \lambda)$$

subject to:

$$0 \leq p^{D1} \leq 100 \quad : \underline{\mu}^{D1}, \overline{\mu}^{D1}$$

For demand D2:

$$\underset{p^{D2}}{\text{Maximize}} \quad p^{D2}(35 - \lambda)$$

subject to:

$$0 \leq p^{D2} \leq 50 \quad : \underline{\mu}^{D2}, \overline{\mu}^{D2}$$

Price-setter:

$$\underset{\lambda}{\text{Minimize}} \quad \lambda \left(p^{D1} + p^{D2} - p^{G1} - p^{G2}\right)$$

Discussion:

What kind of game-theoretic problem is it? Is it a "**non-cooperative**" game? Or a "**cooperative**" one?

Some seminal works on competitive equilibrium

MATHEMATICAL METHODS OF ORGANIZING AND PLANNING PRODUCTION*†

L. V. KANTOROVICH

Leningrad State University

1939

Contents

L. V. Kantorovich, "Mathematical methods of organizing and planning production," *Management Science*, vol. 6, no. 4, pp. 366–422, 1960.

SPATIAL PRICE EQUILIBRIUM AND LINEAR PROGRAMMING

By PAUL A. SAMUELSON*

I.—*Introduction*

Increasingly, modern economic theorists are going beyond the formulation of equilibrium in terms of such marginal equalities as marginal revenue equal to marginal costs or wage rate equal to marginal value product. Instead they are reverting to an earlier and more fundamental aspect of a maximum position: namely, that from the top of a hill, whether or not it is locally flat, all movements are downward.

P. A. Samuelson, "Spatial price equilibrium and linear programming," *American Economic Review*, vol. 42, no. 3, pp. 283–303, 1952.

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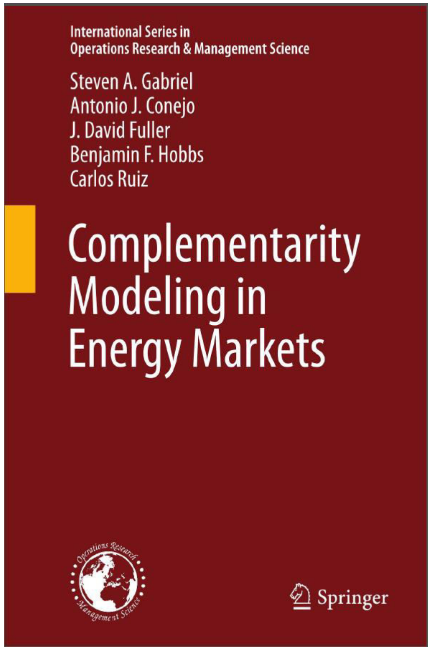
EXISTENCE OF AN EQUILIBRIUM FOR A COMPETITIVE ECONOMY

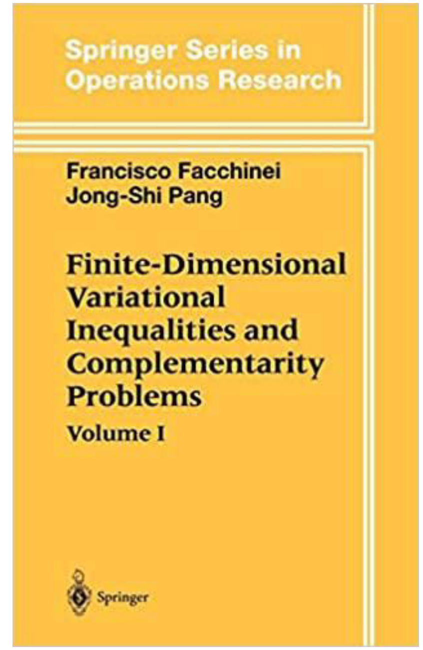
BY KENNETH J. ARROW AND GERARD DEBREU[1]

A. Wald has presented a model of production and a model of exchange and proofs of the existence of an equilibrium for each of them. Here proofs of the existence of an equilibrium are given for an *integrated* model of production, exchange and consumption. In addition the assumptions made on the technologies of producers and the tastes of consumers are significantly weaker than Wald's. Finally a simplification of the structure of the proofs has been made possible through use of the concept of an abstract economy, a generalization of that of a game.

K. J. Arrow and G. Debreu, "Existence of an equilibrium for a competitive economy," *Econometrica*, vol. 22, no. 3, pp. 265–290,1954.

Relevant books and courses!





- Prof. Steven Gabriel's yearly short course at NTNU, "**Introduction Course in Complementarity Models and Equilibrium**": https://www.ntnu.edu/studies/courses/I%C3%988806#tab=omEmnet

- Prof. Uday Shanbhag's invited 5-day course at DTU in 2019. All video lectures are publicly available here: https://www.youtube.com/watch?v=PYXIzmxW53k&list=PLKLR7D59yU0fuZTH5wjgov31D3DXta_l-

Market clearing as an equilibrium problem

- If a solution to our equilibrium problem exists, it will be a "Nash equilibrium point", i.e.,


John Nash

No market participant can increase its profit by deviating unilaterally from the equilibrium solution!

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Discussion:

- Is our market-clearing problem a "Nash equilibrium" (NE) problem?
- Or, is it a "generalized" Nash equilibrium (GNE)?
- What is the difference of NE and GNE? Which one is more appealing?

Market clearing as an equilibrium problem

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No market participant can increase its profit by deviating unilaterally from the equilibrium solution!

Recall the first question:
How to make sure all market participants are satisfied with the outcome of market-clearing optimization problem, and would not deviate from it?

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Recall the first question:
How to make sure all market participants are satisfied with the outcome of market-clearing optimization problem, and would not deviate from it?

And new questions:
- Which problem should we solve to clear the market (optimization or equilibrium)?
- How to solve an equilibrium problem?

Market clearing: optimization vs equilibrium!

Equilibrium

For each generator:

Maximize profit

subject to production limits

For each demand:

Maximize utility

subject to consumption limits

Price-setter's problem

Optimization

Maximize market's social welfare

subject to:

- Production limits of generators
- Consumption limits of demands
- Power balance

How to solve the equilibrium problem?

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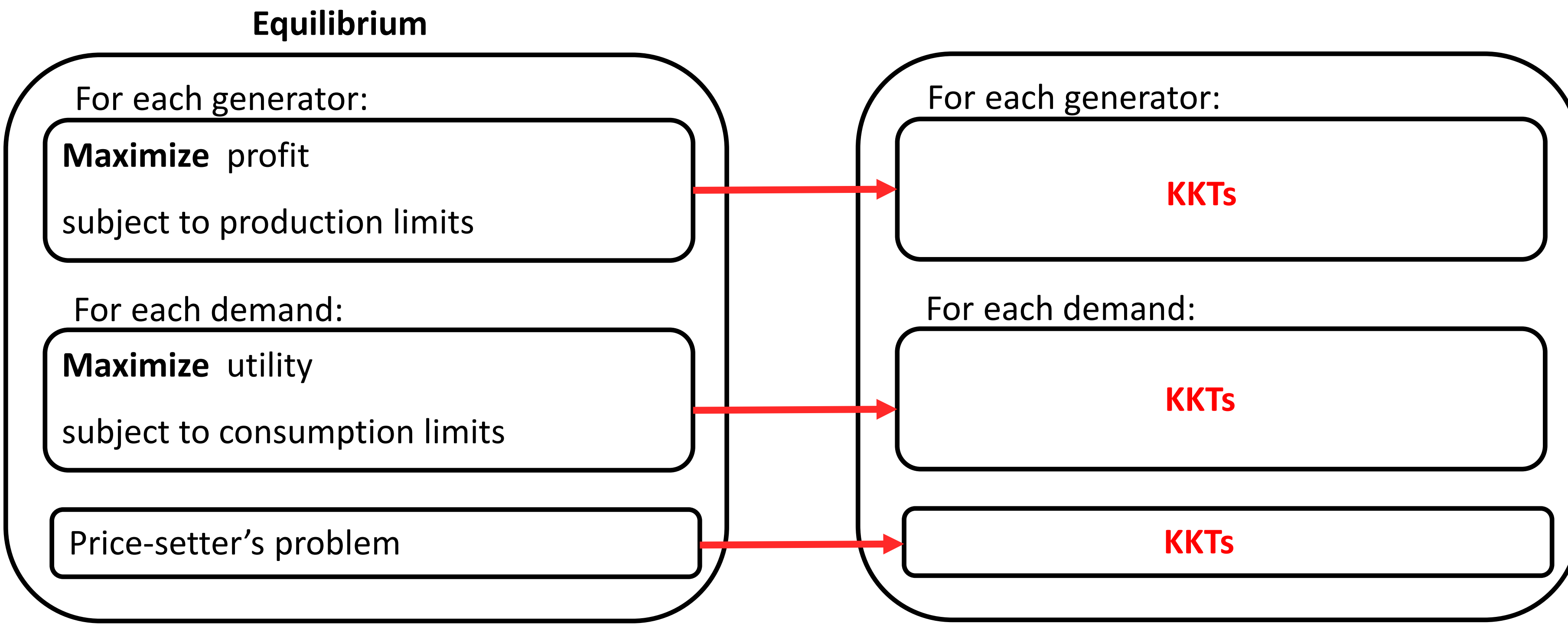
Replace each optimization problem within the equilibrium problem by its equivalent Karush-Kuhn-Tucker (KKT) optimality conditions! Recall that these conditions are a collection of equality and inequality conditions without any objective function!

How to solve the equilibrium problem?



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Price-setter's problem

For each generator:

KKTs

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KKTs

Mixed complementarity problem (MCP)

It is straightforward to solve!

Market clearing: optimization vs equilibrium!

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subject to:

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Let's check again the equilibrium and optimization problems above!

Market clearing: optimization vs equilibrium!

Equilibrium

For each generator:

Maximize profit

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For each demand:

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subject to consumption limits

Price-setter's problem

→ MCP

Optimization

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subject to:

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- Consumption limits of demands
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→ KKTs

Market clearing: optimization vs equilibrium!

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Maximize market's social welfare

subject to:

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→ KKTs

Do the MCP (obtained from equilibrium) and the KKTs (obtained from optimization) include **identical** conditions?

If so, the equilibrium and optimization problems above are "**equivalent**", i.e., any solution to the equilibrium problem is also a solution to the optimization problem and vice versa.

Market clearing as an equilibrium problem

$$\underset{p^{G1}}{\text{Maximize}} \quad p^{G1}(\lambda - 12)$$
subject to:
$$0 \le p^{G1} \le 100 \quad : \underline{\mu}^{G1}, \overline{\mu}^{G1}$$

$$\underset{p^{G2}}{\text{Maximize}} \quad p^{G2}(\lambda - 20)$$
subject to:
$$0 \le p^{G2} \le 80 \quad : \underline{\mu}^{G2}, \overline{\mu}^{G2}$$

$$\underset{p^{D1}}{\text{Maximize}} \quad p^{D1}(40 - \lambda)$$
subject to:
$$0 \le p^{D1} \le 100 \quad : \underline{\mu}^{D1}, \overline{\mu}^{D1}$$

$$\underset{p^{D2}}{\text{Maximize}} \quad p^{D2}(35 - \lambda)$$
subject to:
$$0 \le p^{D2} \le 50 \quad : \underline{\mu}^{D2}, \overline{\mu}^{D2}$$

$$\underset{\lambda}{\text{Minimize}} \quad \lambda\left(p^{D1} + p^{D2} - p^{G1} - p^{G2}\right)$$

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subject to:
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KKTs →

$$12 - \lambda - \underline{\mu}^{G1} + \overline{\mu}^{G1} = 0$$
$$0 \leq p^{G1} \perp \underline{\mu}^{G1} \geq 0$$
$$0 \leq (100 - p^{G1}) \perp \overline{\mu}^{G1} \geq 0$$

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KKTs →

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$$\underset{\lambda}{\text{Minimize}} \quad \lambda\,(p^{D1} + p^{D2} - p^{G1} - p^{G2})$$

→

$$p^{D1} + p^{D2} - p^{G1} - p^{G2} = 0$$

Mixed complementarity problem (MCP)

It is straightforward to solve!

Market clearing as an equilibrium problem

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$$p^{\mathrm{D1}} + p^{\mathrm{D2}} - p^{\mathrm{G1}} - p^{\mathrm{G2}} = 0$$

Mixed complementarity problem (MCP)

Market clearing as an equilibrium problem

Question:

Is this MCP identical to the KKTs of the market-clearing optimization problem? Let's check!

$$12 - \lambda - \underline{\mu}^{\mathrm{G1}} + \overline{\mu}^{\mathrm{G1}} = 0$$
$$0 \leq p^{\mathrm{G1}} \perp \underline{\mu}^{\mathrm{G1}} \geq 0$$
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Answer: ?

Optimization

$$\underset{p^{G1},p^{G2},p^{D1},p^{D2}}{\text{Maximize}} \quad SW = [40p^{D1} + 35p^{D2}] - [12p^{G1} + 20p^{G2}] \tag{1a}$$

subject to:

$$0 \leq p^{D1} \leq 100 \quad : \underline{\mu}^{D1}, \overline{\mu}^{D1} \tag{1b}$$

$$0 \leq p^{D2} \leq 50 \quad : \underline{\mu}^{D2}, \overline{\mu}^{D2} \tag{1c}$$

$$0 \leq p^{G1} \leq 100 \quad : \underline{\mu}^{G1}, \overline{\mu}^{G1} \tag{1d}$$

$$0 \leq p^{G2} \leq 80 \quad : \underline{\mu}^{G2}, \overline{\mu}^{G2} \tag{1e}$$

$$p^{D1} + p^{D2} - p^{G1} - p^{G2} = 0 \quad : \lambda \tag{1f}$$

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Answer: Yes!

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KKTs →

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Conclusions so far!

- The equilibrium and optimization forms of the market-clearing problem are equivalent, because their corresponding KKT conditions are identical!

- Both equilibrium and optimization forms of the market-clearing problem obtain the "Nash equilibrium solution", i.e., no market player desires to deviate from the market-clearing outcomes!

Form 1: Market clearing as an optimization problem

$$\underset{p_g^{\mathrm{G}},\ p_d^{\mathrm{D}},\ \theta_n}{\text{Maximize}} \quad \sum_d U_d\ p_d^{\mathrm{D}} - \sum_g C_g\ p_g^{\mathrm{G}}$$

subject to:

$$0 \leq p_d^{\mathrm{D}} \leq \overline{P}_d^{\mathrm{D}} \quad : \underline{\mu}_d^{\mathrm{D}}, \overline{\mu}_d^{\mathrm{D}} \quad \forall d$$

$$0 \leq p_g^{\mathrm{G}} \leq \overline{P}_g^{\mathrm{G}} \quad : \underline{\mu}_g^{\mathrm{G}}, \overline{\mu}_g^{\mathrm{G}} \quad \forall g$$

$$\sum_{d \in \Psi_n} p_d^{\mathrm{D}} + \sum_{m \in \Omega_n} B_{n,m}(\theta_n - \theta_m) - \sum_{g \in \Psi_n} p_g^{\mathrm{G}} = 0 \quad : \lambda_n \quad \forall n$$

$$-F_{n,m} \leq B_{n,m}(\theta_n - \theta_m) \leq F_{n,m} \quad : \underline{\eta}_{n,m}, \overline{\eta}_{n,m} \quad \forall n, \forall m \in \Omega_n$$

$$\theta_{(n=ref)} = 0 \quad : \gamma$$

Form 2: Market clearing as an equilibrium problem

Each generator:

$$\underset{p_g^{\mathrm{G}}}{\text{Maximize}} \quad p_g^{\mathrm{G}}(\lambda_{n:g\in\Psi_n} - C_g)$$

subject to:

$$0 \leq p_g^{\mathrm{G}} \leq \overline{P}_g^{\mathrm{G}} \quad : \underline{\mu}_g^{\mathrm{G}}, \overline{\mu}_g^{\mathrm{G}}$$

Each elastic demand:

$$\underset{p_d^{\mathrm{D}}}{\text{Maximize}} \quad p_d^{\mathrm{D}}(U_d - \lambda_{n:d\in\Psi_n})$$

subject to:

$$0 \leq p_d^{\mathrm{D}} \leq \overline{P}_d^{\mathrm{D}} \quad : \underline{\mu}_d^{\mathrm{D}}, \overline{\mu}_d^{\mathrm{D}}$$

Transmission owner as a spatial arbitrager (it buys power at a bus and sells it back at another one):

$$\underset{\theta_n}{\text{Maximize}} \quad \sum_{n,(m\in\Omega_n)} \lambda_n[B_{n,m}(\theta_m - \theta_n)]$$

subject to:

$$-F_{n,m} \leq B_{n,m}(\theta_n - \theta_m) \leq F_{n,m} \quad : \underline{\eta}_{n,m}, \overline{\eta}_{n,m} \quad \forall n, \forall m \in \Omega_n$$

Price-setter:

$$\underset{\lambda_n}{\text{Minimize}} \quad \sum_n \lambda_n \left(\sum_{d\in\Psi_n} p_d^{\mathrm{D}} + \sum_{m\in\Omega_n} B_{n,m}(\theta_n - \theta_m) - \sum_{g\in\Psi_n} p_g^{\mathrm{G}} \right)$$

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Transmission owner as a spatial arbitrager (it buys power at a bus and sells it back at another one):

$$\underset{\theta_n}{\text{Maximize}} \quad \sum_{n,(m\in\Omega_n)} \lambda_n[B_{n,m}(\theta_m - \theta_n)]$$

This term is known as "congestion rent"!

subject to:

$$-F_{n,m} \leq B_{n,m}(\theta_n - \theta_m) \leq F_{n,m} \quad : \underline{\eta}_{n,m}, \overline{\eta}_{n,m} \quad \forall n, \forall m \in \Omega_n$$

Price-setter:

$$\underset{\lambda_n}{\text{Minimize}} \quad \sum_n \lambda_n \left(\sum_{d\in\Psi_n} p_d^{\mathrm{D}} + \sum_{m\in\Omega_n} B_{n,m}(\theta_n - \theta_m) - \sum_{g\in\Psi_n} p_g^{\mathrm{G}} \right)$$

Closer look into congestion rent

Transmission owner as a spatial arbitrager (it buys power at a bus and sells it back at another one):

$$\underset{\theta_n}{\text{Maximize}} \quad \sum_{n,(m \in \Omega_n)} \lambda_n [B_{n,m}(\theta_m - \theta_n)]$$

Let's investigate if this objective function is correct!

subject to:

$$-F_{n,m} \leq B_{n,m}(\theta_n - \theta_m) \leq F_{n,m} \quad : \underline{\eta}_{n,m}, \overline{\eta}_{n,m} \quad \forall n, \forall m \in \Omega_n$$

Closer look into congestion rent

Transmission owner as a spatial arbitrager (it buys power at a bus and sells it back at another one):

$$\underset{\theta_n}{\text{Maximize}} \quad \sum_{n,(m \in \Omega_n)} \lambda_n [B_{n,m}(\theta_m - \theta_n)]$$

Question: What are "financial transmission rights (FTRs)"?

subject to:

$$-F_{n,m} \leq B_{n,m}(\theta_n - \theta_m) \leq F_{n,m} \quad : \underline{\eta}_{n,m}, \overline{\eta}_{n,m} \quad \forall n, \forall m \in \Omega_n$$

Closer look into congestion rent

Transmission owner as a spatial arbitrager (it buys power at a bus and sells it back at another one):

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Question: What are "financial transmission rights (FTRs)"?

Have you heard of "financial storage rights"? What about "physical storage rights"?

Form 2: Market clearing as an equilibrium problem

Each generator:

$$\underset{p_g^{\mathrm{G}}}{\text{Maximize}} \quad p_g^{\mathrm{G}}(\lambda_{n:g\in\Psi_n} - C_g)$$

subject to:

$$0 \leq p_g^{\mathrm{G}} \leq \overline{P}_g^{\mathrm{G}} \quad : \underline{\mu}_g^{\mathrm{G}}, \overline{\mu}_g^{\mathrm{G}}$$

Each elastic demand:

$$\underset{p_d^{\mathrm{D}}}{\text{Maximize}} \quad p_d^{\mathrm{D}}(U_d - \lambda_{n:d\in\Psi_n})$$

subject to:

$$0 \leq p_d^{\mathrm{D}} \leq \overline{P}_d^{\mathrm{D}} \quad : \underline{\mu}_d^{\mathrm{D}}, \overline{\mu}_d^{\mathrm{D}}$$

Transmission owner as a spatial arbitrager (it buys power at a bus and sells it back at another one):

$$\underset{\theta_n}{\text{Maximize}} \quad \sum_{n,(m\in\Omega_n)} \lambda_n[B_{n,m}(\theta_m - \theta_n)]$$

subject to:

$$-F_{n,m} \leq B_{n,m}(\theta_n - \theta_m) \leq F_{n,m} \quad : \underline{\eta}_{n,m}, \overline{\eta}_{n,m} \quad \forall n, \forall m \in \Omega_n$$

Price-setter:

$$\underset{\lambda_n}{\text{Minimize}} \quad \sum_n \lambda_n \left(\sum_{d\in\Psi_n} p_d^{\mathrm{D}} + \sum_{m\in\Omega_n} B_{n,m}(\theta_n - \theta_m) - \sum_{g\in\Psi_n} p_g^{\mathrm{G}} \right)$$

Form 3: Market clearing as an MCP

$$-U_d + \lambda_{n:d\in\Psi_n} - \underline{\mu}_d^{\mathrm{D}} + \overline{\mu}_d^{\mathrm{D}} = 0 \quad \forall d$$

$$C_g - \lambda_{n:g\in\Psi_n} - \underline{\mu}_g^{\mathrm{G}} + \overline{\mu}_g^{\mathrm{G}} = 0 \quad \forall g$$

$$\sum_{m\in\Omega_n} B_{n,m}(\lambda_n - \lambda_m + \overline{\eta}_{n,m} - \overline{\eta}_{m,n} - \underline{\eta}_{n,m} + \underline{\eta}_{m,n}) + (\gamma)_{n=ref} = 0 \quad \forall n$$

$$\sum_{d\in\Psi_n} p_d^{\mathrm{D}} + \sum_{m\in\Omega_n} B_{n,m}(\theta_n - \theta_m) - \sum_{g\in\Psi_n} p_g^{\mathrm{G}} = 0 \quad \forall n$$

$$\theta_{(n=ref)} = 0$$

$$0 \le p_g^{\mathrm{G}} \perp \underline{\mu}_g^{\mathrm{G}} \ge 0 \quad \forall g$$

$$0 \le [\overline{P}_g^{\mathrm{G}} - p_g^{\mathrm{G}}] \perp \overline{\mu}_g^{\mathrm{G}} \ge 0 \quad \forall g$$

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$$0 \le [F_{n,m} + B_{n,m}(\theta_n - \theta_m)] \perp \underline{\eta}_{n,m} \ge 0 \quad \forall n, \forall m \in \Omega_n$$

$$0 \le [F_{n,m} - B_{n,m}(\theta_n - \theta_m)] \perp \overline{\eta}_{n,m} \ge 0 \quad \forall n, \forall m \in \Omega_n$$

Conclusion

All three forms of the market-clearing problem, i.e.,

- optimization
- equilibrium
- MCP

are equivalent!

Jalal Kazempour

Exercises 1-4

1- Which form of the market-clearing problem (optimization or equilibrium) is more appealing to market operators?

2- Consider an equilibrium form of the market-clearing problem. Is it possible to solve it iteratively without deriving KKTs? If so, what are the pros and cons?

Guide: Consider an iterative mechanism, in which the market operator fixes a set of initial prices, and then each market participant makes her own dispatch decisions accordingly. Based on the participants' dispatch decisions, the market operator checks whether nodal power balance conditions hold or not. If not, the operator "systematically" adjusts those prices and disseminates the updated prices among participants. This can be continued until there is no demand-supply mismatch. If interested, read about "Walrasian auction" and its "tâtonnement process", which indeed requires a decomposition technique (e.g., Lagrangian relaxation or ADMM).

3- For a given Nash equilibrium (NE) problem, how to mathematically identify that an equivalent optimization problem exists?

Guide: check chapter 4 of the book by S. Gabriel et al. (available on DTU Inside) and learn about "Principle of Symmetry", referring to the symmetry of Jacobian matrix. Search how we can derive the Jacobian matrix of a game. You can also check Theorem 1.3.1 of the book by F. Facchinei and J.-S. Pang.

4- Investigate how the solution existence and the solution uniqueness for a Nash equilibrium (NE) can be mathematically proven.

Guide: For uniqueness, search about "monotonicity" property of a game. How can we ensure a game is strongly monotone by checking the Jacobian matrix? You can also read about "degree theory".

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Four desirable properties of market-clearing mechanisms are:

Market efficiency	**Incentive compatibility**
Cost recovery	**Revenue adequacy**

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- In an incentive-compatible market, if the production cost of a generator is $12/MWh, the **dominant** (most profitable) strategy for that generator is to offer "trustfully" at $12/MWh, not at any different price!

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- In other words, no market player desires to exercise "market power" by behaving "strategically", i.e., by submitting "strategic" offers.

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- This property is also known as "**individual rationality**" (although based on some definition in the literature there might be slight differences).

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- In other words, the total payment that the market operator receives from demands is always <u>higher than or equal to</u> her total payment to generators, curtailed loads, transmission operator, etc.
- As a specific status of revenue adequacy, the market is "**budget balance**" if the market operator has neither financial deficit nor excess.

Discussion

Question:

Is there any market-clearing mechanism ensuring all four properties?

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Answer:
No!

- Based on Hurwicz theorem (also known as "impossibility theorem") [1]-[2], no mechanism is capable of achieving all those properties at the same time!

- We have to find a "trade-off" among properties achieved and those lost.

[1] L. Hurwicz, ``On Informationally Decentralized Systems'' in *Decision and Organization*, edited by C.B. McGuire and R. Radner, Amsterdam, 1972.

[2] R. Myerson and M. A. Satterthwaite, "Efficient mechanisms for bilateral trading," *Journal of Economic Theory*, vol. 28, pp. 265–281, 1983.

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Recall that market-clearing models determine the nodal market-clearing prices (LMPs) based on dual variable of nodal power balance equalities. Let’s refer this pricing method to as “LMP-based market mechanism”.

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For each generator:

Maximize profit

subject to production limits

For each demand:

Maximize utility

subject to consumption limits

Power balance

Optimization

Maximize market's social welfare

subject to:

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Question:

Which properties are ensured in the LMP-based market mechanism?

Properties of LMP-based market mechanism

“Incentive compatibility” ensured?

“Market efficiency” ensured?

“Revenue adequacy” ensured?

“Cost recovery” ensured?

Properties of LMP-based market mechanism

"Incentive compatibility" ensured?

No! A market player (the so-called "strategic" player) may exercise "market power" by not trustfully offering in terms of price and/or quantity!

"Market efficiency" ensured?

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Properties of LMP-based market mechanism

“Incentive compatibility” ensured?

No! A market player (the so-called “strategic” player) may exercise “market power” by not trustfully offering in terms of price and/or quantity!

“Market efficiency” ensured?

No! in the sense that if “market power” is exercised, the market’s social welfare will be decreased.

“Revenue adequacy” ensured?

Yes! Proof as an exercise; see the next slide!

“Cost recovery” ensured?

Yes! Proof as an exercise; see the next slide!

Exercise 5

Provide a mathematical proof that the LMP-based market mechanism ensures "revenue adequacy" and even "budget balance".

Guide:

Step 1- Consider the nodal power balance equality for bus *n*.

Step 2- Multiply each term within the equality of Step 1 by the LMP at that bus.

Step 3- Consider the summation of all equalities obtained in Step 2 for all buses. What does the resulting equality mean?

Exercise 6

Provide a mathematical proof that the LMP-based market mechanism ensures "cost recovery" for all market players.

Guide:

Step 1- Consider the equilibrium form of the market-clearing problem.

Step 2- For each generator's optimization problem, derive the corresponding "strong duality" condition, which enforces the equality of objective function of primal and dual problems at the optimal point. The primal objective function is generator's profit. Check the terms within the dual objective function -- are they all non-negative? If so, what does it mean?

Step 3- Similar to Step 2, investigate the cost recovery for elastic demands and transmission system operator using the equilibrium problem in Step 1.

Exercise 6

Provide a mathematical proof that the LMP-based market mechanism ensures "cost recovery" for all market players.

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Step 1- Consider the equilibrium form of the market-clearing problem.

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Step 3- Similar to Step 2, investigate the cost recovery for elastic demands and transmission system operator using the equilibrium problem in Step 1.

Question: If the lower bound for the generation level of a generator is a positive (non-zero) value, can we still ensure cost recovery?

Thank you!

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