DTU CEE Summer School 2019
Data-Driven Analytics and Optimization for Energy Systems
June 16-21 2019

Distributed Optimization

Lecture 4: Exchange / consensus ADMM

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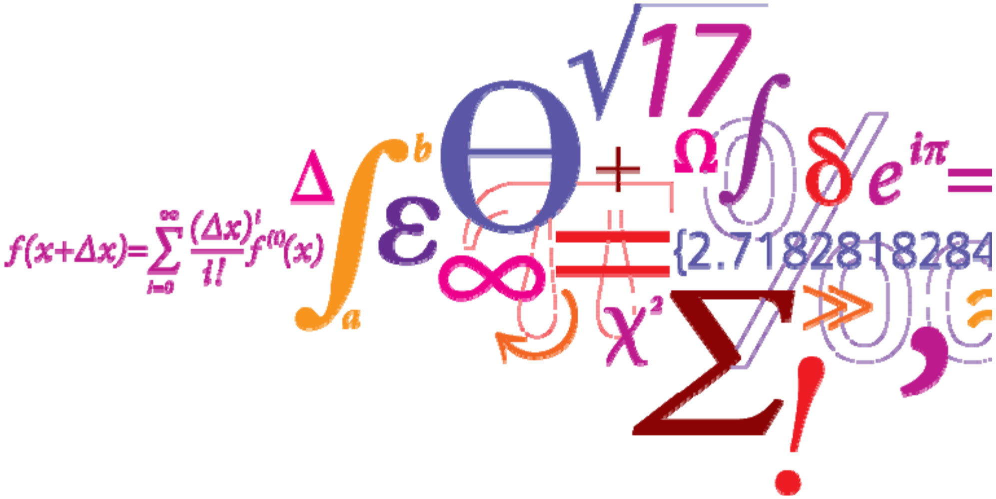
June 21, 2019



DTU Electrical Engineering
Department of Electrical Engineering

Learning objectives

After Lecture 4, you are expected to:

- Have a clearer idea on the applications of (augmented) Lagrangian relaxation and ADMM to energy systems.
- Explain the functioning of Exchange and Consensus ADMM.
- Implement them to illustrative examples.

Example 1: Optimal power flow

Consider a power system with 3 generators with quadratic cost functions and one inelastic demand:

$$\underset{p_g}{\text{Minimize}} \quad \sum_{g=1}^{3} (a_g\, p_g^2 + b_g\, p_g + c_g)$$

Subject to

$$0 \leq p_g \leq P_g^{\max} \qquad \forall g = 1,2,3$$

$$\sum_{g=1}^{3} p_g = D \qquad (\lambda)$$

Example 1: Optimal power flow

Consider a power system with 3 generators with quadratic cost functions and one inelastic demand:

Cost coefficients (constants)

$$\underset{p_g}{\text{Minimize}} \sum_{g=1}^{3} (a_g p_g^2 + b_g p_g + c_g)$$

Production level of generator *g*

Subject to

Capacity of generator *g*

$$0 \leq p_g \leq P_g^{\max} \qquad \forall g = 1,2,3$$

$$\sum_{g=1}^{3} p_g = D \qquad (\lambda)$$

Load level (constant)

Dual variable

Example 1: Optimal power flow

Consider a power system with 3 generators with quadratic cost functions and one inelastic demand:

$$\underset{p_g}{\text{Minimize}} \quad \sum_{g=1}^{3} (a_g\, p_g^2 + b_g\, p_g + c_g)$$

Subject to

$$0 \leq p_g \leq P_g^{\max} \qquad \forall g = 1{,}2{,}3$$

$$\sum_{g=1}^{3} p_g = D \qquad\qquad (\lambda)$$

Reminder from the last lecture: This problem can be decomposed by relaxing power balance equality (complicating constraint). We can implement Lagrangian relaxation (LR), since the objective function is quadratic (first derivative continuous).

Example 1: Optimal power flow

The exact equivalent problem (but still not decomposed) is a max-min problem:

$$\underset{\lambda}{\text{Maximize}} \left\{ \underset{p_g}{\text{Minimize}} \sum_{g=1}^{3} (a_g\, p_g^2 + b_g\, p_g + c_g) + \lambda \left[D - \sum_{g=1}^{3} p_g \right] \right.$$

Subject to

$$\left. 0 \leq p_g \leq P_g^{\max} \qquad \forall g = 1,2,3 \right\}$$

Example 1: Optimal power flow

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Subject to

$$\left. 0 \le p_g \le P_g^{\max} \qquad \forall g = 1,2,3 \right\}$$

Then, pursuing decomposability, we fix dual variable λ to a given value $\bar{\lambda}$. This yields:

$$\underset{p_g}{\text{Minimize}} \sum_{g=1}^{3}(a_g\, p_g^2 + b_g\, p_g + c_g) + \bar{\lambda} \left[D - \sum_{g=1}^{3} p_g \right]$$

Subject to

$$0 \le p_g \le P_g^{\max} \qquad \forall g = 1,2,3$$

This problem is now decomposed to three subproblems, one per generator!

Example 1: Optimal power flow

Three subproblems, one per generator:

$$\left\{\begin{array}{l} \underset{p_g}{\text{Minimize}} \quad (a_g\, p_g^2 \;+ b_g\, p_g) - \bar{\lambda}\, p_g \\ \qquad\qquad \text{Subject to} \\ \qquad\qquad 0 \leq p_g \leq P_g^{\max} \end{array}\right\} \quad \forall g = 1,2,3.$$

Example 1: Optimal power flow

Three subproblems, one per generator:

$$\left\{ \begin{array}{l} \underset{p_g}{\text{Minimize}} \;\; (a_g\, p_g^2 \; + b_g\, p_g) - \bar{\lambda}\, p_g \\ \qquad\qquad \text{Subject to} \\ \qquad\qquad 0 \leq p_g \leq P_g^{\max} \end{array} \right\} \forall g = 1,2,3.$$

Then, the value of $\bar{\lambda}$ should be updated for the next iteration (e.g., using subgradient method) [1]: Solve subproblems 1, 2 and 3 in iteration v to obtain the values $p_g^{(v)}$:

$$\bar{\lambda}^{(v+1)} \leftarrow \bar{\lambda}^{(v)} + \frac{1}{a+bv} \frac{D - \sum_{g=1}^{3} p_g^{(v)}}{\left| D - \sum_{g=1}^{3} p_g^{(v)} \right|}$$

where a and b are positive constants, e.g., $a = 1$ and $b = 0.1$.

[1] A. J. Conejo, E. Castillo, R. Minguez, and R. Garcia-Bertrand, *Decomposition Techniques in Mathematical Programming: Engineering and Science Applications*. Berlin, Germany: Springer, 2006.

Example 1: Optimal power flow

The illustration of LR operation:

Subproblem 1 for generator 1:

$$\underset{p_1^{(v)}}{\text{Minimize}}\ (b_1\, p_1^{2(v)} + b_1\, p_1^{(v)}) - \bar{\lambda}^{(v)}\, p_1^{(v)}$$

Subject to $0 \leq p_1^{(v)} \leq P_1^{\max}$

Subproblem 2 for generator 2:

$$\underset{p_2^{(v)}}{\text{Minimize}}\ (b_2\, p_2^{2(v)} + b_2\, p_2^{(v)}) - \bar{\lambda}^{(v)}\, p_2^{(v)}$$

Subject to $0 \leq p_2^{(v)} \leq P_2^{\max}$

Subproblem 3 for generator 3:

$$\underset{p_3^{(v)}}{\text{Minimize}}\ (b_3\, p_3^{2(v)} + b_3\, p_3^{(v)}) - \bar{\lambda}^{(v)}\, p_3^{(v)}$$

Subject to $0 \leq p_3^{(v)} \leq P_3^{\max}$

Example 1: Optimal power flow

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Subject to $0 \leq p_3^{(v)} \leq P_3^{\max}$

Coordinator ($\bar{\lambda}$-update)

$$\bar{\lambda}^{(v+1)} \leftarrow \bar{\lambda}^{(v)} + \frac{1}{a+bv}\,\frac{D - \sum_{g=1}^{3} p_g^{(v)}}{\left|D - \sum_{g=1}^{3} p_g^{(v)}\right|}$$

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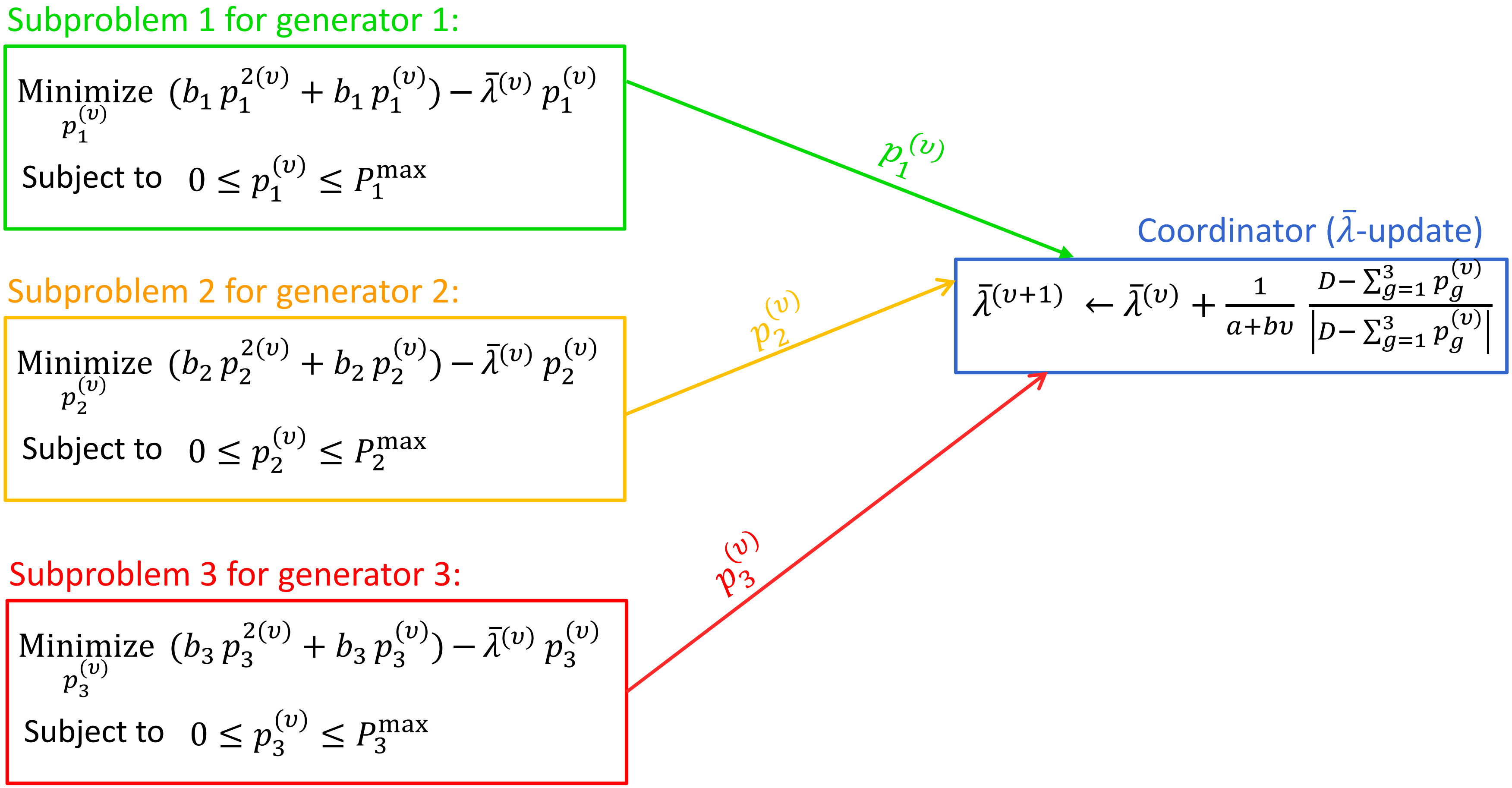

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Subproblem 2 for generator 2:

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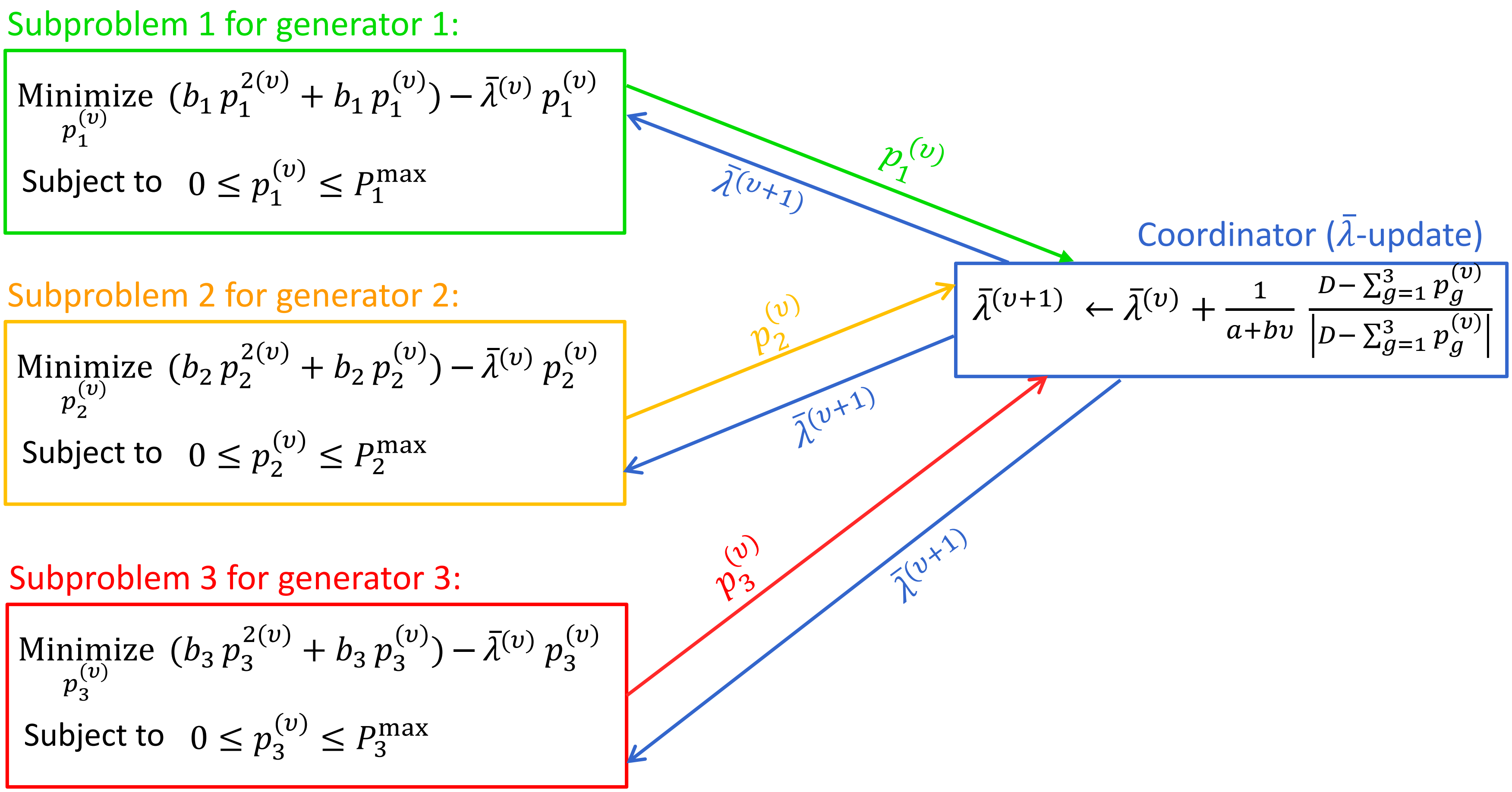

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Subject to $0 \leq p_1^{(v)} \leq P_1^{\max}$

Subproblem 2 for generator 2:

$$\underset{p_2^{(v)}}{\text{Minimize}} \ (b_2\, p_2^{2(v)} + b_2\, p_2^{(v)}) - \bar{\lambda}^{(v)}\, p_2^{(v)}$$

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$$\underset{p_3^{(v)}}{\text{Minimize}} \ (b_3\, p_3^{2(v)} + b_3\, p_3^{(v)}) - \bar{\lambda}^{(v)}\, p_3^{(v)}$$

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$$\bar{\lambda}^{(v+1)} \leftarrow \bar{\lambda}^{(v)} + \frac{1}{a+bv}\,\frac{D-\sum_{g=1}^{3} p_g^{(v)}}{\left|D-\sum_{g=1}^{3} p_g^{(v)}\right|}$$

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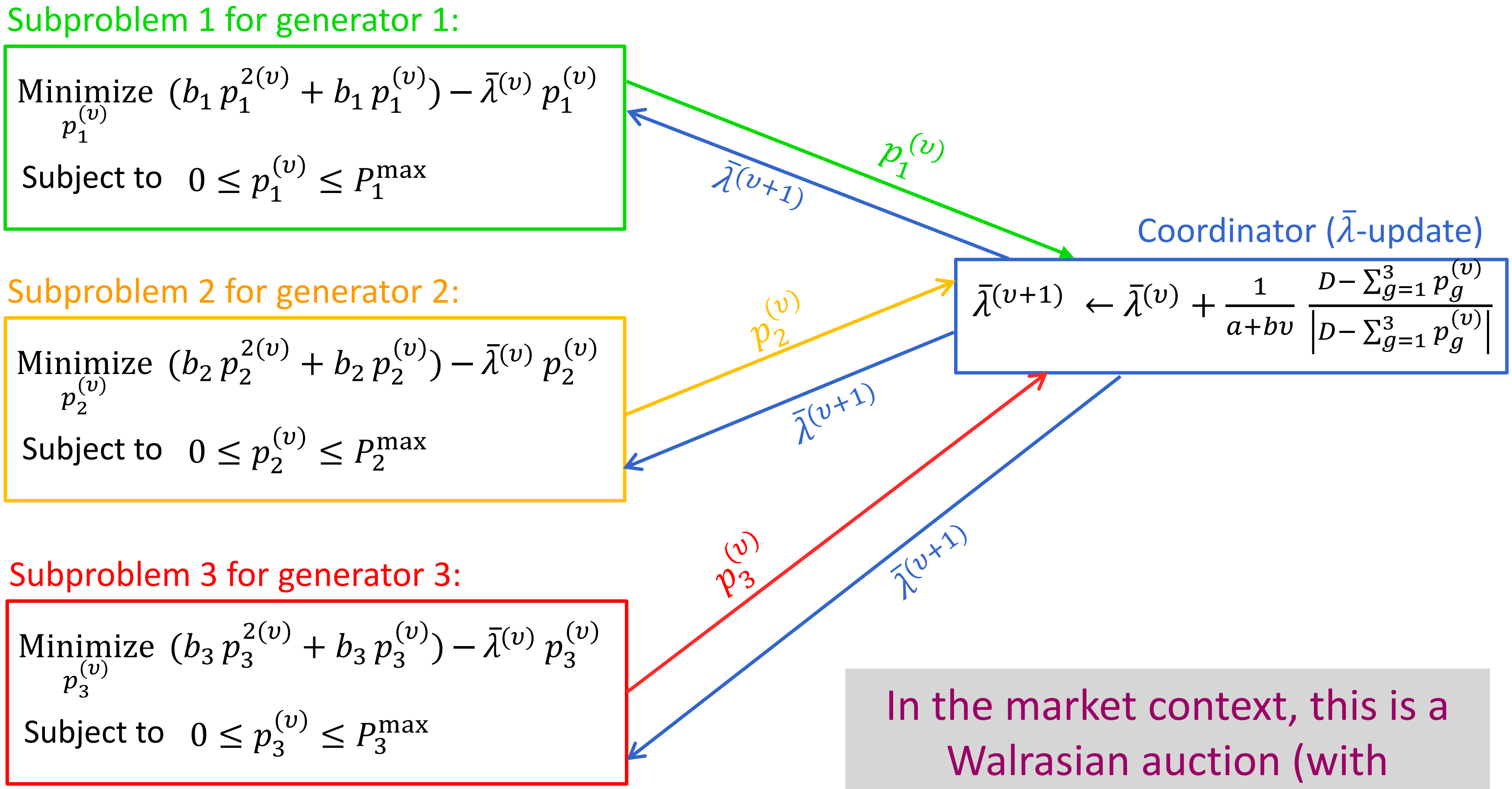
The illustration of LR operation:



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Subject to $\quad 0 \le p_1^{(v)} \le P_1^{\max}$

Subproblem 2 for generator 2:

$$\underset{p_2^{(v)}}{\text{Minimize}} \ (b_2\, p_2^{2(v)} + b_2\, p_2^{(v)}) - \bar{\lambda}^{(v)}\, p_2^{(v)}$$

Subject to $\quad 0 \le p_2^{(v)} \le P_2^{\max}$

Subproblem 3 for generator 3:

$$\underset{p_3^{(v)}}{\text{Minimize}} \ (b_3\, p_3^{2(v)} + b_3\, p_3^{(v)}) - \bar{\lambda}^{(v)}\, p_3^{(v)}$$

Subject to $\quad 0 \le p_3^{(v)} \le P_3^{\max}$

Coordinator ($\bar{\lambda}$-update)

$$\bar{\lambda}^{(v+1)} \leftarrow \bar{\lambda}^{(v)} + \frac{1}{a+bv}\, \frac{D - \sum_{g=1}^{3} p_g^{(v)}}{\left|D - \sum_{g=1}^{3} p_g^{(v)}\right|}$$

In the market context, this is a Walrasian auction (with quadratic offers of generators)!

Example 1: Optimal power flow

Discussion:

How to generalize Example 1 including elastic demands and power transmission system?

Example 2: Market clearing

Consider an electricity market with 3 generators (with linear offers) and one inelastic demand:

$$\underset{p_g}{\text{Minimize}} \quad \sum_{g=1}^{3} C_g \, p_g$$

Subject to

$$0 \leq p_g \leq P_g^{\max} \qquad \forall g = 1,2,3$$

$$\sum_{g=1}^{3} p_g = D \qquad\qquad (\lambda)$$

Example 2: Market clearing

Consider an electricity market with 3 generators (with linear offers) and one inelastic demand:

Offer price of generator *g* (parameter)

$$\underset{p_g}{\text{Minimize}} \sum_{g=1}^{3} C_g p_g$$

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Subject to

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$$\sum_{g=1}^{3} p_g = D \qquad (\lambda)$$

Reminder from the last lecture: We cannot implement Lagrangian relaxation (LR), since the objective function is linear (first derivative is a constant). The alternative is to implement augmented Lagrangian relaxation (ALR).

Example 2: Market clearing

The exact equivalent problem (but still not decomposed) is the following max-min problem. We have added a weighted quadratic penalty term (whose value is equal to zero in the optimal point) to make the first derivative of objective function continuous:

$$\underset{\lambda}{\text{Maximize}} \left\{ \underset{p_g}{\text{Minimize}} \sum_{g=1}^{3} C_g\, p_g + \lambda \left[D - \sum_{g=1}^{3} p_g \right] + \frac{\gamma}{2} \left\| D - \sum_{g=1}^{3} p_g \right\|^2 \right.$$

Subject to

$$\left. 0 \le p_g \le P_g^{\max} \qquad \forall g = 1,2,3 \right\}$$

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Subject to

$$0 \leq p_g \leq P_g^{\max} \qquad \forall g = 1,2,3 \Bigg\}$$

Then, pursuing decomposability, we fix dual variable λ to a given value $\bar{\lambda}$. This yields:

$$\underset{p_g}{\text{Minimize}} \sum_{g=1}^{3} C_g \, p_g + \bar{\lambda} \left[D - \sum_{g=1}^{3} p_g \right] + \frac{\gamma}{2} \left\| D - \sum_{g=1}^{3} p_g \right\|^2$$

Subject to

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Subject to

$$0 \le p_g \le P_g^{\max} \qquad \forall g = 1{,}2{,}3$$

Still not decomposed! Why?

Example 2: Market clearing

We use “alternating direction method of multipliers (ADMM)” to solve ALR in a decomposed manner. Three subproblems, one per generator:

$$\left.\begin{cases}\underset{p_g}{\text{Minimize}}\ C_g\, p_g - \bar{\lambda}\, p_g + \dfrac{\gamma}{2}\left\| D - p_g - \displaystyle\sum_{\substack{i=1\\ i\neq g}}^{3} p_i \right\|^2 \\ \text{Subject to} \\ 0 \leq p_g \leq P_g^{\max} \end{cases}\right\} \quad \forall g = 1,2,3.$$

Example 2: Market clearing

We use "alternating direction method of multipliers (ADMM)" to solve ALR in a decomposed manner. Three subproblems, one per generator:

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Then, the value of $\bar{\lambda}$ should be updated for the next iteration [1]: Solve subproblems 1, 2 and 3 in iteration υ to obtain the values $p_g^{(\upsilon)}$:

$$\bar{\lambda}^{(\upsilon)} \leftarrow \bar{\lambda}^{(\upsilon-1)} + \gamma\left(D - \sum_{g=1}^{3} p_g^{(\upsilon)}\right)$$

[1] S. Boyd, N. Parikh, E. Chu, B. Peleato, and J. Eckstein, "Distributed optimization and statistical learning via the alternating direction method of multipliers," *Foundations and Trends in Machine Learning*, vol. 3, no. 1, pp. 1-122, Jan. 2011.

Example 2: Market clearing

The illustration of ADMM operation:

Subproblem 1 for generator 1:

$$\underset{p_1^{(v)}}{\text{Minimize}}\ C_1\, p_1^{(v)} - \bar{\lambda}^{(v)}\ p_1^{(v)} + \frac{\gamma}{2}\left\| D - p_1^{(v)} - p_2^{(v-1)} - p_3^{(v-1)} \right\|^2$$

Subject to $0 \leq p_1^{(v)} \leq P_1^{\max}$

Subproblem 2 for generator 2:

$$\underset{p_2^{(v)}}{\text{Minimize}}\ C_2\, p_2^{(v)} - \bar{\lambda}^{(v)}\ p_2^{(v)} + \frac{\gamma}{2}\left\| D - p_2^{(v)} - p_1^{(v-1)} - p_3^{(v-1)} \right\|^2$$

Subject to $0 \leq p_2^{(v)} \leq P_2^{\max}$

Subproblem 3 for generator 3:

$$\underset{p_3^{(v)}}{\text{Minimize}}\ C_3\, p_3^{(v)} - \bar{\lambda}^{(v)}\ p_3^{(v)} + \frac{\gamma}{2}\left\| D - p_3^{(v)} - p_1^{(v-1)} - p_2^{(v-1)} \right\|^2$$

Subject to $0 \leq p_3^{(v)} \leq P_3^{\max}$

Example 2: Market clearing

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Subject to $0 \leq p_3^{(v)} \leq P_3^{\max}$

$$\bar{\lambda}^{(v+1)} \leftarrow \bar{\lambda}^{(v)} + \gamma\left(D - p_1^{(v)} - p_2^{(v)} - p_3^{(v)}\right)$$

Coordinator ($\bar{\lambda}$-update)

Example 2: Market clearing

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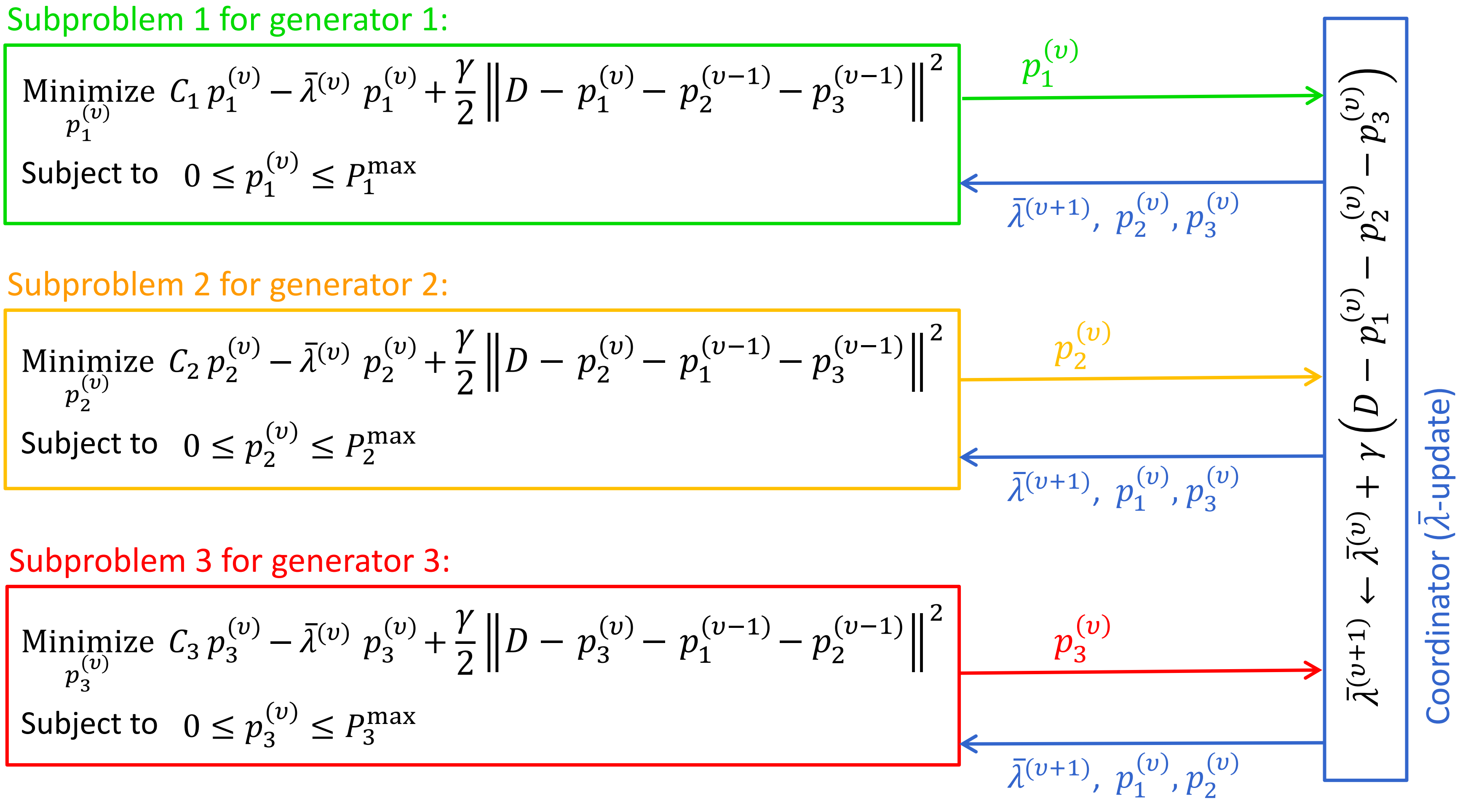

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Subject to $0 \leq p_1^{(v)} \leq P_1^{\max}$

$p_1^{(v)}$

$\bar{\lambda}^{(v+1)},\ p_2^{(v)}, p_3^{(v)}$

Subproblem 2 for generator 2:

$$\underset{p_2^{(v)}}{\text{Minimize}}\ C_2\, p_2^{(v)} - \bar{\lambda}^{(v)}\, p_2^{(v)} + \frac{\gamma}{2}\left\| D - p_2^{(v)} - p_1^{(v-1)} - p_3^{(v-1)} \right\|^2$$

Subject to $0 \leq p_2^{(v)} \leq P_2^{\max}$

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Subproblem 3 for generator 3:

$$\underset{p_3^{(v)}}{\text{Minimize}}\ C_3\, p_3^{(v)} - \bar{\lambda}^{(v)}\, p_3^{(v)} + \frac{\gamma}{2}\left\| D - p_3^{(v)} - p_1^{(v-1)} - p_2^{(v-1)} \right\|^2$$

Subject to $0 \leq p_3^{(v)} \leq P_3^{\max}$

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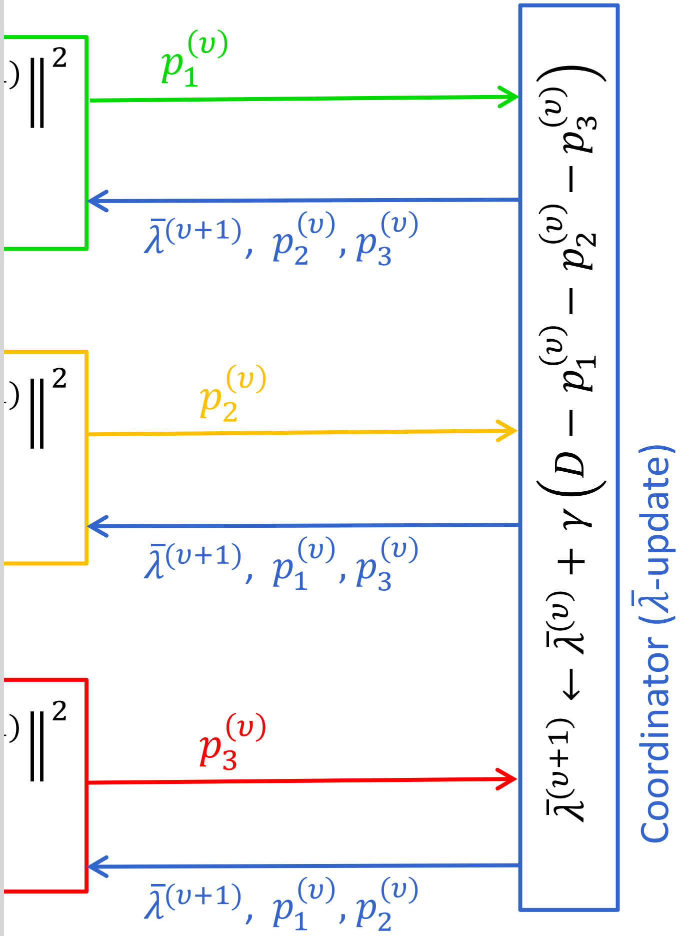
$$\bar{\lambda}^{(v+1)} \leftarrow \bar{\lambda}^{(v)} + \gamma\left(D - p_1^{(v)} - p_2^{(v)} - p_3^{(v)}\right)$$

Coordinator ($\bar{\lambda}$-update)

Example 2: Market clearing

The illustration of ADMM operation:

In each iteration, each generator needs to know the dispatch of other generators in the previous iteration to be able to solve its own subproblem. From market perspective, does this make sense? Is this a Walrasian auction?



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Coordinator ($\bar{\lambda}$-update)

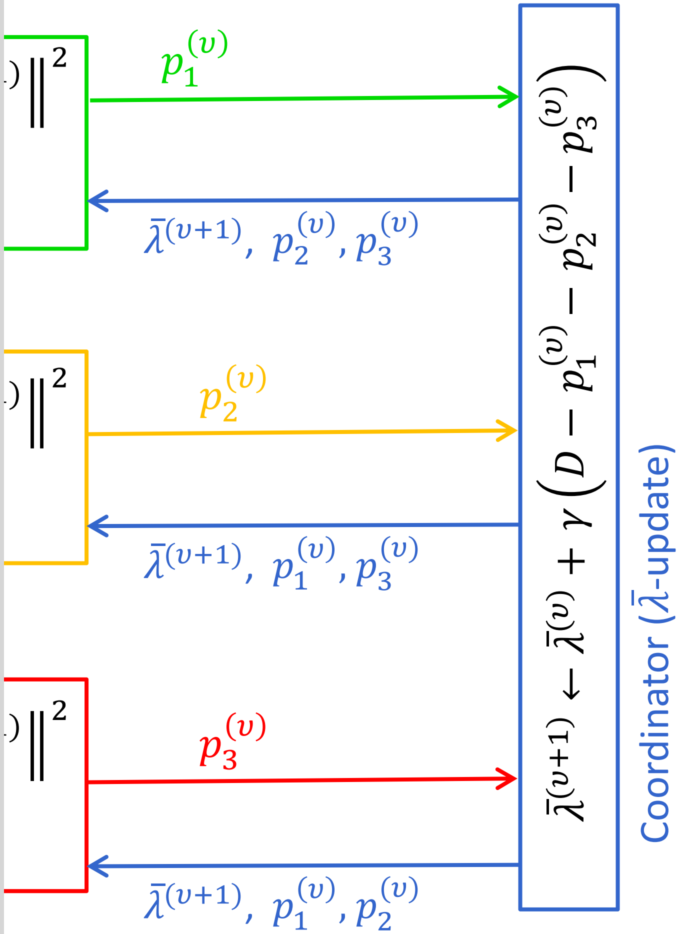
Example 2: Market clearing

The illustration of ADMM operation:

In each iteration, each generator needs to know the dispatch of other generators in the previous iteration to be able to solve its own subproblem. From market perspective, does this make sense? Is this a Walrasian auction?

- we will talk about "**Exchange ADMM**", which resolves this issue, i.e., each generator does not need any information of other generators.

- We will also talk about "**Consensus ADMM**"!



Main Reference

S. Boyd, N. Parikh, E. Chu, B. Peleato, and J. Eckstein, "Distributed optimization and statistical learning via the alternating direction method of multipliers," *Foundations and Trends in Machine Learning*, vol. 3, no. 1, pp. 1-122, Jan. 2011.

Exchange ADMM

Exchange ADMM

Consider the following compact form of market-clearing (optimal exchange) problem with inelastic demands:

$$\underset{x_i}{\text{Minimize}} \sum_{i=1}^{N} f_i(x_i)$$

Subject to

$$h_i(x_i) \leq A_i \qquad \forall i = 1, \dots, N$$

$$\sum_{i=1}^{N} x_i = D \qquad (\lambda)$$

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$$\sum_{i=1}^{N} x_i = D \qquad (\lambda)$$

It is straightforward to write a similar problem with elastic demands. In that case, the right-hand side of complicating constraint will be zero.

Exchange ADMM

ADMM-based solution:

$$\underset{x_i}{\text{Minimize}} \sum_{i=1}^{N} f_i(x_i) + \bar{\lambda} \left(\sum_{i=1}^{N} x_i - D \right) + \frac{\gamma}{2} \left\| \sum_{i=1}^{N} x_i - D \right\|^2$$

Subject to

$$h_i(x_i) \leq A_i \qquad \forall i = 1, \ldots, N$$

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ADMM-based solution:

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Subject to

$$h_i(x_i) \leq A_i \qquad \forall i = 1, \dots, N$$

Each subproblem:

$$\left\{ \begin{array}{l} \underset{x_i^{(v)}}{\text{Minimize}} \quad f_i(x_i^{(v)}) + \bar{\lambda}^{(v-1)} x_i^{(v)} + \frac{\gamma}{2} \left\| x_i^{(v)} + \sum_{\substack{j=1 \\ j \neq i}}^{N} x_j^{(v-1)} - D \right\|^2 \\ \text{Subject to} \\ h_i(x_i^{(v)}) \leq A_i \end{array} \right\} \forall i$$

and $\bar{\lambda}^{(v)} \leftarrow \bar{\lambda}^{(v-1)} + \gamma \left(\sum_{i=1}^{N} x_i^{(v)} - D \right)$

Exchange ADMM

ADMM-based solution:

$$\underset{x_i}{\text{Minimize}} \sum_{i=1}^{N} f_i(x_i) + \bar{\lambda} \left(\sum_{i=1}^{N} x_i - D\right) + \frac{\gamma}{2} \left\| \sum_{i=1}^{N} x_i - D \right\|^2$$

Subject to

$$h_i(x_i) \leq A_i \qquad \forall i = 1, \dots, N$$

Each subproblem:

$$\left\{ \begin{array}{l} \underset{x_i^{(v)}}{\text{Minimize}} \quad f_i(x_i^{(v)}) + \bar{\lambda}^{(v-1)} x_i^{(v)} + \frac{\gamma}{2} \left\| x_i^{(v)} + \sum_{\substack{j=1 \\ j \neq i}}^{N} x_j^{(v-1)} - D \right\|^2 \\ \text{Subject to} \\ h_i(x_i^{(v)}) \leq A_i \end{array} \right\} \forall i$$

and $\bar{\lambda}^{(v)} \leftarrow \bar{\lambda}^{(v-1)} + \gamma \left(\boxed{\sum_{i=1}^{N} x_i^{(v)}} - D \right)$

$= N\, \bar{x}^{(v)}$

where $\bar{x}^{(v)}$ is the **average** production in iteration v!

Exchange ADMM

ADMM-based solution:

$$\underset{x_i}{\text{Minimize}} \sum_{i=1}^{N} f_i(x_i) + \bar{\lambda} \left(\sum_{i=1}^{N} x_i - D\right) + \frac{\gamma}{2}\left\|\sum_{i=1}^{N} x_i - D\right\|^2$$

Subject to

$$h_i(x_i) \le A_i \qquad \forall i = 1, \dots, N$$

$= N\,\bar{x}^{(v-1)} - x_i^{(v-1)}$

Each subproblem:

$$\left\{\begin{array}{l} \underset{x_i^{(v)}}{\text{Minimize}} \quad f_i(x_i^{(v)}) + \bar{\lambda}^{(v-1)} x_i^{(v)} + \frac{\gamma}{2}\left\| x_i^{(v)} + \boxed{\sum_{\substack{j=1\\ j\neq i}}^{N} x_j^{(v-1)}} - D\right\|^2 \\ \text{Subject to} \\ h_i(x_i^{(v)}) \le A_i \end{array}\right\} \forall i$$

$= N\,\bar{x}^{(v)}$

where $\bar{x}^{(v)}$ is the **average** production in iteration v!

and $\bar{\lambda}^{(v)} \leftarrow \bar{\lambda}^{(v-1)} + \gamma\left(\boxed{\sum_{i=1}^{N} x_i^{(v)}} - D\right)$

Exchange ADMM

Important observation: Each agent does not need to know the dispatch information of other agents in details. The "average" dispatch is sufficient!

$= N\,\bar{x}^{(v-1)} - x_i^{(v-1)}$

Each subproblem:

$$\left\{\begin{array}{l}\underset{x_i^{(v)}}{\text{Minimize}} \quad f_i(x_i^{(v)}) + \bar{\lambda}^{(v-1)} x_i^{(v)} + \frac{\gamma}{2}\left\| x_i^{(v)} + \boxed{\sum_{\substack{j=1\\ j\neq i}}^{N} x_j^{(v-1)}} - D \right\|^2 \\ \text{Subject to} \\ h_i(x_i^{(v)}) \le A_i \end{array}\right\} \quad \forall i$$

$= N\,\bar{x}^{(v)}$

where $\bar{x}^{(v)}$ is the **average** production in iteration v!

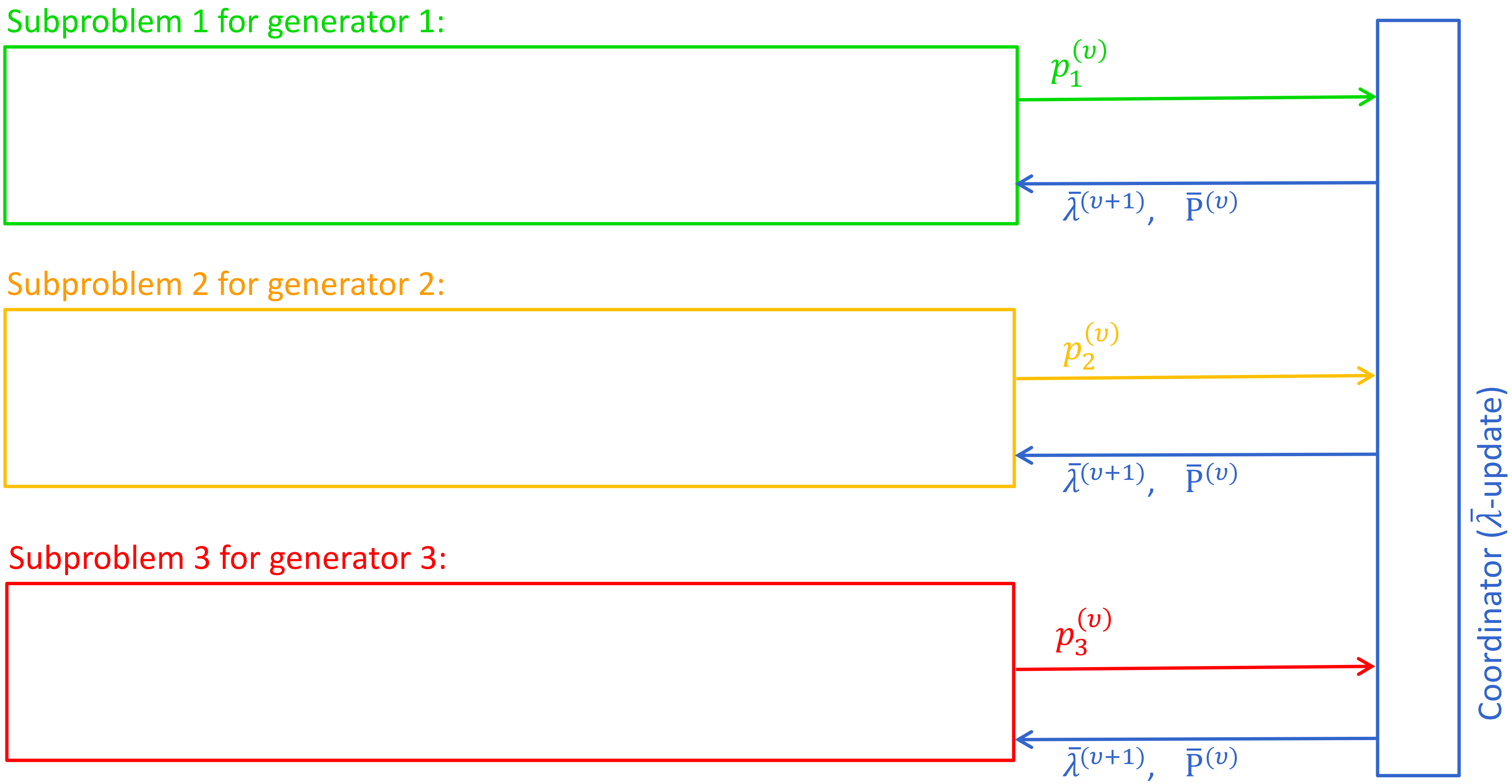
and $\bar{\lambda}^{(v)} \leftarrow \bar{\lambda}^{(v-1)} + \gamma\left(\boxed{\sum_{i=1}^{N} x_i^{(v)}} - D\right)$

An example of a market with 3 generators

An example of a market with 3 generators

The illustration of "Exchange ADMM" operation:

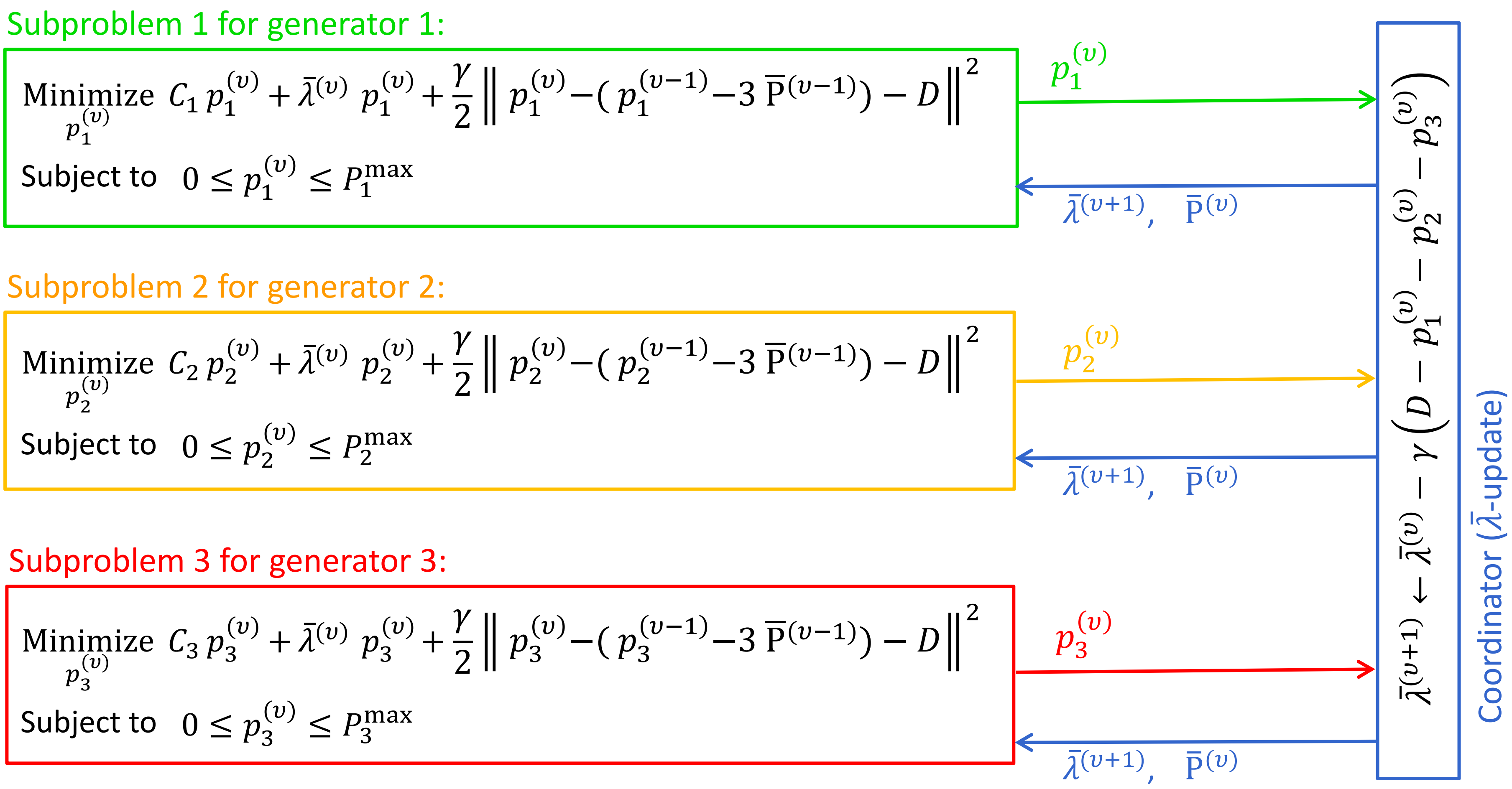
Note: $\overline{\mathrm{P}}^{(\upsilon)}$ is the average production dispatch of three generators in iteration υ.



An example of a market with 3 generators

The illustration of “Exchange ADMM” operation:

Note: $\overline{\mathrm{P}}^{(v)}$ is the average production dispatch of three generators in iteration v.



Subproblem 1 for generator 1:

$$\underset{p_1^{(v)}}{\text{Minimize}} \;\; C_1\, p_1^{(v)} + \bar{\lambda}^{(v)}\, p_1^{(v)} + \frac{\gamma}{2}\left\| p_1^{(v)} - (\, p_1^{(v-1)} - 3\,\overline{\mathrm{P}}^{(v-1)}) - D \right\|^2$$

$$\text{Subject to} \quad 0 \le p_1^{(v)} \le P_1^{\max}$$

Subproblem 2 for generator 2:

$$\underset{p_2^{(v)}}{\text{Minimize}} \;\; C_2\, p_2^{(v)} + \bar{\lambda}^{(v)}\, p_2^{(v)} + \frac{\gamma}{2}\left\| p_2^{(v)} - (\, p_2^{(v-1)} - 3\,\overline{\mathrm{P}}^{(v-1)}) - D \right\|^2$$

$$\text{Subject to} \quad 0 \le p_2^{(v)} \le P_2^{\max}$$

Subproblem 3 for generator 3:

$$\underset{p_3^{(v)}}{\text{Minimize}} \;\; C_3\, p_3^{(v)} + \bar{\lambda}^{(v)}\, p_3^{(v)} + \frac{\gamma}{2}\left\| p_3^{(v)} - (\, p_3^{(v-1)} - 3\,\overline{\mathrm{P}}^{(v-1)}) - D \right\|^2$$

$$\text{Subject to} \quad 0 \le p_3^{(v)} \le P_3^{\max}$$

$p_1^{(v)}$

$\bar{\lambda}^{(v+1)}, \quad \overline{\mathrm{P}}^{(v)}$

$p_2^{(v)}$

$\bar{\lambda}^{(v+1)}, \quad \overline{\mathrm{P}}^{(v)}$

$p_3^{(v)}$

$\bar{\lambda}^{(v+1)}, \quad \overline{\mathrm{P}}^{(v)}$

$$\bar{\lambda}^{(v+1)} \leftarrow \bar{\lambda}^{(v)} - \gamma\left(D - p_1^{(v)} - p_2^{(v)} - p_3^{(v)}\right)$$

Coordinator ($\bar{\lambda}$-update)

Consensus ADMM

Consensus ADMM

Consider the following problem, including agents $i = 1, \dots, N$, but a single global variable x. This problem is called the "*global consensus problem*", since all agents should agree on x.

$$\underset{x}{\text{Minimize}} \sum_{i=1}^{N} f_i(x)$$

Subject to

$$h_i(x) \leq A_i \qquad \forall i = 1, \dots, N$$

Consensus ADMM

Consider the following problem, including agents $i = 1, \dots, N$, but a single global variable x. This problem is called the "*global consensus problem*", since all agents should agree on x.

$$\underset{x}{\text{Minimize}} \sum_{i=1}^{N} f_i(x)$$

Subject to

$$h_i(x) \leq A_i \qquad \forall i = 1, \dots, N$$

This problem can be rewritten as follows using auxiliary variable z:

$$\underset{x_i, z}{\text{Minimize}} \sum_{i=1}^{N} f_i(x_i)$$

Subject to

$$h_i(x_i) \leq A_i \qquad \forall i = 1, \dots, N$$

$$x_i = z \qquad (\lambda_i) \qquad \forall i = 1, \dots, N$$

Consensus ADMM

x-update subproblem for each agent i:

z-update subproblem (based on given values for $x_i^{(v)}$):

$\bar{\lambda}$-update for each agent i:

Consensus ADMM

x-update subproblem for each agent i:

$$\left.\begin{cases} \underset{x_i^{(v)}}{\text{Minimize}} \quad f_i(x_i^{(v)}) + \overline{\lambda}_i^{(v-1)} x_i^{(v)} + \frac{\gamma}{2} \left\| x_i^{(v)} - z^{(v-1)} \right\|^2 \\ \text{Subject to} \\ h_i(x_i^{(v)}) \leq A_i \end{cases}\right\} \forall i$$

z-update subproblem (based on given values for $x_i^{(v)}$):

$\overline{\lambda}$-update for each agent i:

Consensus ADMM

x-update subproblem for each agent i:

$$\left\{\begin{aligned} &\underset{x_i^{(\upsilon)}}{\text{Minimize}} \quad f_i(x_i^{(\upsilon)}) + \bar{\lambda}_i^{(\upsilon-1)} x_i^{(\upsilon)} + \frac{\gamma}{2}\left\| x_i^{(\upsilon)} - z^{(\upsilon-1)} \right\|^2 \\ &\text{Subject to} \\ &h_i(x_i^{(\upsilon)}) \le A_i \end{aligned}\right\} \forall i$$

z-update subproblem (based on given values for $x_i^{(\upsilon)}$):

$$\underset{z^{(\upsilon)}}{\text{Minimize}} \; -\sum_{i=1}^{N} \bar{\lambda}_i^{(\upsilon-1)} z^{(\upsilon)} + \frac{\gamma}{2}\sum_{i=1}^{N}\left\| x_i^{(\upsilon)} - z^{(\upsilon)} \right\|^2$$

$\bar{\lambda}$-update for each agent i:

Consensus ADMM

x-update subproblem for each agent i:

$$\left\{\begin{array}{l} \underset{x_i^{(v)}}{\text{Minimize}} \quad f_i(x_i^{(v)}) + \overline{\lambda}_i^{(v-1)} x_i^{(v)} + \frac{\gamma}{2}\left\| x_i^{(v)} - z^{(v-1)} \right\|^2 \\ \text{Subject to} \\ h_i(x_i^{(v)}) \leq A_i \end{array}\right\} \quad \forall i$$

z-update subproblem (based on given values for $x_i^{(v)}$):

$$\underset{z^{(v)}}{\text{Minimize}} \; -\sum_{i=1}^{N} \overline{\lambda}_i^{(v-1)} z^{(v)} + \frac{\gamma}{2}\sum_{i=1}^{N}\left\| x_i^{(v)} - z^{(v)} \right\|^2$$

$\overline{\lambda}$-update for each agent i:

$$\overline{\lambda}_i^{(v)} \leftarrow \overline{\lambda}_i^{(v-1)} + \gamma\left(x_i^{(v)} - z^{(v)}\right) \qquad \forall i$$

Consensus ADMM

z-update subproblem (based on given values for $x_i^{(v)}$):

$$\underset{z^{(v)}}{\text{Minimize}} \; -\sum_{i=1}^{N} \bar{\lambda}_i^{(v-1)} z^{(v)} + \frac{\gamma}{2}\sum_{i=1}^{N} \left\| x_i^{(v)} - z^{(v)} \right\|^2$$

Let’s first focus on this single subproblem!

Consensus ADMM

$$z^{(v)} = \frac{1}{N}\sum_{i=1}^{N}\left(x_i^{(v)} + \frac{1}{\gamma}\bar{\lambda}_i^{(v-1)}\right)$$

This constraint-free optimization problem can be easily solved. This yields:

z-update subproblem (based on given values for $x_i^{(v)}$):

$$\underset{z^{(v)}}{\text{Minimize}} \; -\sum_{i=1}^{N}\bar{\lambda}_i^{(v-1)} z^{(v)} + \frac{\gamma}{2}\sum_{i=1}^{N}\left\|x_i^{(v)} - z^{(v)}\right\|^2$$

Consensus ADMM

$$z^{(v)} = \frac{1}{N} \sum_{i=1}^{N} \left(x_i^{(v)} + \frac{1}{\gamma} \overline{\lambda}_i^{(v-1)} \right)$$

This constraint-free optimization problem can be easily solved. This yields:

z-update subproblem (based on given values for $x_i^{(v)}$):

$$\underset{z^{(v)}}{\text{Minimize}} \; -\sum_{i=1}^{N} \overline{\lambda}_i^{(v-1)} z^{(v)} + \frac{\gamma}{2} \sum_{i=1}^{N} \left\| x_i^{(v)} - z^{(v)} \right\|^2$$

The z-update subproblem is not an optimization anymore! This is called as "*central collector*" or "*fusion center*".

Consensus ADMM: updated formulation

x-update subproblem for each agent i:

$$\left\{\begin{array}{l} \underset{x_i^{(\upsilon)}}{\text{Minimize}} \quad f_i(x_i^{(\upsilon)}) + \overline{\lambda}_i^{(\upsilon-1)} x_i^{(\upsilon)} + \frac{\gamma}{2}\left\|x_i^{(\upsilon)} - z^{(\upsilon-1)}\right\|^2 \\ \text{Subject to} \\ h_i(x_i^{(\upsilon)}) \leq A_i \end{array}\right\} \quad \forall i$$

z-update subproblem (based on given values for $x_i^{(\upsilon)}$):

$$z^{(\upsilon)} = \frac{1}{N}\sum_{i=1}^{N}\left(x_i^{(\upsilon)} + \frac{1}{\gamma}\overline{\lambda}_i^{(\upsilon-1)}\right)$$

$\overline{\lambda}$-update for each agent i:

$$\overline{\lambda}_i^{(\upsilon)} \leftarrow \overline{\lambda}_i^{(\upsilon-1)} + \gamma\left(x_i^{(\upsilon)} - z^{(\upsilon)}\right) \quad \forall i$$

Consensus ADMM: updated formulation

x-update subproblem for each agent i:

$$\left\{\begin{array}{l} \underset{x_i^{(v)}}{\text{Minimize}} \quad f_i(x_i^{(v)}) + \overline{\lambda}_i^{(v-1)} x_i^{(v)} + \frac{\gamma}{2}\left\| x_i^{(v)} - z^{(v-1)} \right\|^2 \\ \text{Subject to} \\ h_i(x_i^{(v)}) \leq A_i \end{array}\right\} \forall i$$

z-update subproblem (based on given values for $x_i^{(v)}$):

$$z^{(v)} = \frac{1}{N}\sum_{i=1}^{N}\left(x_i^{(v)} + \frac{1}{\gamma}\overline{\lambda}_i^{(v-1)}\right)$$

This algorithm can be even further simplified!

$\overline{\lambda}$-update for each agent i:

$$\overline{\lambda}_i^{(v)} \leftarrow \overline{\lambda}_i^{(v-1)} + \gamma\left(x_i^{(v)} - z^{(v)}\right) \quad \forall i$$

Consensus ADMM: updated formulation

x-update subproblem for each agent i:

$$\left\{\begin{array}{l} \underset{x_i^{(v)}}{\text{Minimize}} \quad f_i(x_i^{(v)}) + \overline{\lambda}_i^{(v-1)} x_i^{(v)} + \frac{\gamma}{2} \left\| x_i^{(v)} - z^{(v-1)} \right\|^2 \\ \text{Subject to} \\ h_i(x_i^{(v)}) \leq A_i \end{array}\right\} \quad \forall i$$

z-update subproblem (based on given values for $x_i^{(v)}$):

$$z^{(v)} = \frac{1}{N} \sum_{i=1}^{N} \left(x_i^{(v)} + \frac{1}{\gamma} \overline{\lambda}_i^{(v-1)} \right) \quad \Longrightarrow \quad z^{(v)} = \bar{x}^{(v)} + \frac{1}{\gamma} \bar{\lambda}^{(v-1)}$$

Written in an average form over *i=1,…,N*

$\overline{\lambda}$-update for each agent i:

$$\overline{\lambda}_i^{(v)} \leftarrow \overline{\lambda}_i^{(v-1)} + \gamma \left(x_i^{(v)} - z^{(v)} \right) \quad \forall i$$

Consensus ADMM: updated formulation

x-update subproblem for each agent i:

$$\left\{\begin{array}{l} \underset{x_i^{(v)}}{\text{Minimize}} \quad f_i(x_i^{(v)}) + \overline{\lambda}_i^{(v-1)} x_i^{(v)} + \frac{\gamma}{2}\left\| x_i^{(v)} - z^{(v-1)} \right\|^2 \\ \text{Subject to} \\ h_i(x_i^{(v)}) \leq A_i \end{array}\right\} \forall i$$

z-update subproblem (based on given values for $x_i^{(v)}$):

$$z^{(v)} = \frac{1}{N}\sum_{i=1}^{N}\left(x_i^{(v)} + \frac{1}{\gamma}\overline{\lambda}_i^{(v-1)}\right) \quad \Longrightarrow \quad z^{(v)} = \bar{x}^{(v)} + \frac{1}{\gamma}\bar{\lambda}^{(v-1)}$$

Written in an average form over *i=1,…,N*

$\bar{\lambda}$-update for each agent i:

$$\overline{\lambda}_i^{(v)} \leftarrow \overline{\lambda}_i^{(v-1)} + \gamma\left(x_i^{(v)} - z^{(v)}\right) \quad \forall i \quad \Longrightarrow \quad \bar{\lambda}^{(v)} = \bar{\lambda}^{(v-1)} + \gamma\left(\bar{x}^{(v)} - z^{(v)}\right)$$

Written in an average form over *i=1,…,N*

Consensus ADMM: updated formulation

x-update subproblem for each agent i:

$$\left\{\begin{aligned} &\underset{x_i^{(v)}}{\text{Minimize}} \quad f_i(x_i^{(v)}) + \overline{\lambda_i}^{(v-1)} x_i^{(v)} + \frac{\gamma}{2}\left\|x_i^{(v)} - z^{(v-1)}\right\|^2 \\ &\text{Subject to} \\ &h_i(x_i^{(v)}) \leq A_i \end{aligned}\right\} \forall i$$

z-update subproblem (based on given values for $x_i^{(v)}$):

?

$$\left\{\begin{aligned} z^{(v)} &= \bar{x}^{(v)} + \frac{1}{\gamma}\bar{\lambda}^{(v-1)} \\ \bar{\lambda}^{(v)} &= \bar{\lambda}^{(v-1)} + \gamma\left(\bar{x}^{(v)} - z^{(v)}\right) \end{aligned}\right.$$

Consensus ADMM: updated formulation

x-update subproblem for each agent i:

$$\left\{ \begin{array}{l} \underset{x_i^{(v)}}{\text{Minimize}} \quad f_i(x_i^{(v)}) + \overline{\lambda}_i^{(v-1)} x_i^{(v)} + \frac{\gamma}{2} \left\| x_i^{(v)} - z^{(v-1)} \right\|^2 \\ \text{Subject to} \\ h_i(x_i^{(v)}) \leq A_i \end{array} \right\} \forall i$$

z-update subproblem (based on given values for $x_i^{(v)}$):

$$\left. \begin{array}{l} \bar{\lambda}^{(v)} = 0 \\ z^{(v)} = \bar{x}^{(v)} \end{array} \right\} \Longleftarrow \left\{ \begin{array}{l} z^{(v)} = \bar{x}^{(v)} + \frac{1}{\gamma} \bar{\lambda}^{(v-1)} \\ \bar{\lambda}^{(v)} = \bar{\lambda}^{(v-1)} + \gamma\left(\bar{x}^{(v)} - z^{(v)}\right) \end{array} \right.$$

Consensus ADMM: final form

x-update subproblem for each agent i:

$$\left.\begin{cases} \underset{x_i^{(v)}}{\text{Minimize}} \quad f_i(x_i^{(v)}) + \overline{\lambda}_i^{(v-1)} x_i^{(v)} + \frac{\gamma}{2} \left\| x_i^{(v)} - \bar{x}^{(v-1)} \right\|^2 \\ \text{Subject to} \\ h_i(x_i^{(v)}) \leq A_i \end{cases}\right\} \quad \forall i$$

$\overline{\lambda}$-update for each agent i:

$$\overline{\lambda}_i^{(v)} \leftarrow \overline{\lambda}_i^{(v-1)} + \gamma \left(x_i^{(v)} - \bar{x}^{(v)} \right) \quad \forall i$$

In each iteration v, each agent i independently solves its own subproblem to determine the value of global variable $x_i^{(v)}$, while panelized based on its distance to the average value of global variable among all agents obtained in the previous iteration!

Exercise 3

Consider a two-stage stochastic programming problem, e.g., a two-settlement market-clearing problem with day ahead (DA) and real time (RT) stages:

$$\underset{p_g^{\mathrm{DA}},\ p_{g,\omega}^{\mathrm{RT}},\ p_{d,\omega}^{\mathrm{shed}}}{\text{Minimize}} \quad \mathbf{Cost}^{\mathrm{DA}}(p_g^{\mathrm{DA}}) + \mathbb{E}_{\omega}\left[\mathbf{Cost}^{\mathrm{RT}}(p_{g,\omega}^{\mathrm{RT}},\ p_{d,\omega}^{\mathrm{shed}})\right]$$

subject to:

$$\mathbf{f}(p_g^{\mathrm{DA}}) \leq 0$$

$$\mathbf{g}(p_{g,\omega}^{\mathrm{RT}},\ p_{d,\omega}^{\mathrm{shed}}) \leq 0 \qquad \forall \omega$$

$$\mathbf{h}(p_g^{\mathrm{DA}},\ p_{g,\omega}^{\mathrm{RT}},\ p_{d,\omega}^{\mathrm{shed}}) \leq 0 \qquad \forall \omega$$

Can this problem be solved by “consensus ADMM”? If so, how?

Guide: Think of relaxing the non-anticipativity conditions, and to have one subproblem per scenario.

Thanks for your attention!

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