DTU CEE Summer School 2019

Data-Driven Analytics and Optimization for Energy Systems

June 16-21 2019

Distributed Optimization

Lecture 3: (Augmented) Lagrangian Relaxation

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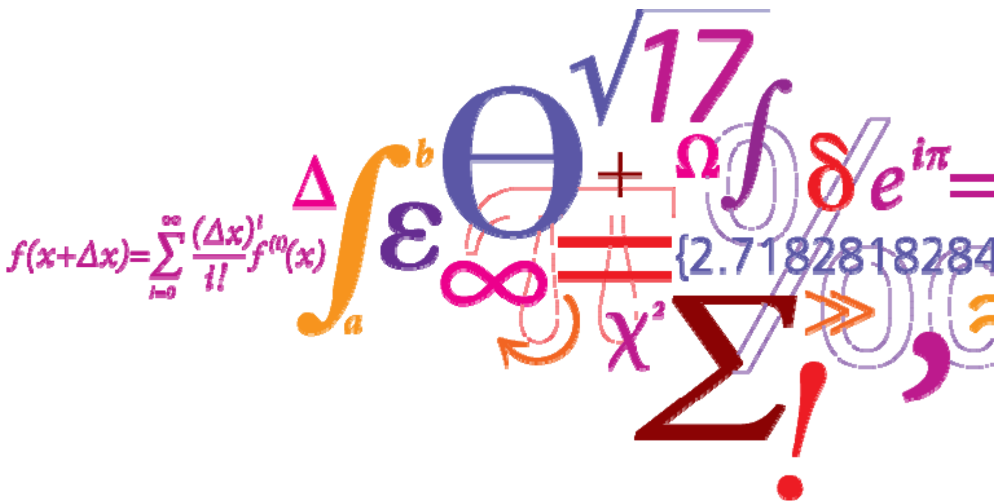
June 21, 2019



DTU Electrical Engineering

Department of Electrical Engineering

Learning objectives

After Lecture 3, you are expected to be able to:

- Explain the functioning of Lagrangian relaxation (LR), augmented Lagrangian relaxation (ALR), and alternating direction method of multipliers (ADMM)
- Implement them to an illustrative example

Decomposition techniques

Applicable to optimization problems with complicating constraints:

- Lagrangian relaxation (LR)

  In the literature, this technique has also been known as *standard* or *conventional* LR (or *dual decomposition*)!

- Augmented Lagrangian relaxation (ALR)
  - ❑ Auxiliary problem principle (APP)
  - ❑ Alternating direction method of multipliers (ADMM)

- Dantzig-Wolfe decomposition

- …

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- Dantzig-Wolfe decomposition
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Will not be covered in this course!

Potential applications in power systems

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- **Market-clearing problem**
  - ✓ **Complicating constraints**: balance equalities and ramping limits of generators
  - ✓ If relaxed, the original problem decomposes by agent (and by hour)

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- **Market-clearing problem**
  - ✓ **Complicating constraints**: balance equalities and ramping limits of generators
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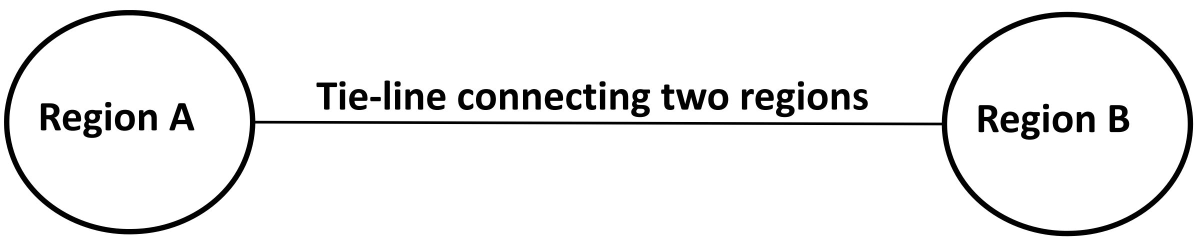
- **Multi-regional market-clearing (or unit commitment) problem, e.g., in case of pan-European electricity market**



  - ✓ **Complicating constraints**: tie-line constraints (power flow, and tie-line capacity)
  - ✓ If relaxed, the original problem decomposes by region. This way, the operator of each region only solves its own market-clearing problem (the so-called *distributed* market-clearing problem)

Main references

- A. J. Conejo, E. Castillo, R. Minguez, and R. Garcia-Bertrand, *Decomposition Techniques in Mathematical Programming: Engineering and Science Applications*. Berlin, Germany: Springer, 2006.

- S. Boyd, N. Parikh, E. Chu, B. Peleato, and J. Eckstein, “Distributed optimization and statistical learning via the alternating direction method of multipliers,” *Foundations and Trends in Machine Learning*, vol. 3, no. 1, pp. 1-122, Jan. 2011.

Lagrangian relaxation (LR)

Background:

- The theory of LR (and also augmented LR) was firstly developed for problems with continuous variables, and functions (objective function and constraints) with first derivatives continuous.

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- The theory of LR (and also augmented LR) was firstly developed for problems with continuous variables, and functions (objective function and constraints) with first derivatives continuous.

- However, the theory has been used in problems with binary variables (like unit commitment problems) with success.

Lagrangian relaxation (LR)

Background:

- LR works efficiently if the number of complicating constraints is relatively low, and it is OK to have binary variables in the formulation.

- LR was extensively used in the 90's to solve unit commitment problems (complicating constrains are just balance constraints and ramping constraints).

Lagrangian relaxation (LR)

<u>Key point</u>

In case of LR:

In addition to convexity, the objective function of the original (non-decomposed) problem needs to be smooth (continuous first derivatives), e.g., quadratic. If this objective function is linear, the LR procedure does <u>not</u> necessarily converge!

Lagrangian relaxation (LR)

<u>Key point</u>

In case of LR:

In addition to convexity, the objective function of the original (non-decomposed) problem needs to be smooth (continuous first derivatives), e.g., quadratic. If this objective function is linear, the LR procedure does <u>not</u> necessarily converge!

- Alternative solution technique for problems with <u>linear</u> objective function is **<u>augmented LR</u>**.

Lagrangian relaxation (LR)

For unit commitment (and also market clearing) problems:

- LR (for problems with quadratic objective function)
- ALR (for problems with either quadratic or linear objective function)

Both have been extensively and very successfully used in the literature, though unit commitment problem is non-convex (due to binary variables).

LR: Mathematical procedure

Consider the following optimization problem:

$$\underset{x_i}{\text{Minimize}} \sum_{i=1}^{I} f_i(x_i)$$

Subject to

$$g_i(x_i) = A_i \qquad \forall i$$

$$h_i(x_i) \leq B_i \qquad \forall i$$

$$\sum_{i=1}^{I} c_i(x_i) = M \qquad (\lambda)$$

$$\sum_{i=1}^{I} d_i(x_i) \leq N \qquad (\mu)$$

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Is it decomposable?

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Dual variables
(Lagrangian multipliers)

LR: Mathematical procedure

The original problem is equivalent to its ***Lagrangian dual problem*** (a max-min problem):

$$\underset{\lambda,\mu}{\text{Maximize}} \left\{ \underset{x_i}{\text{Minimize}} \sum_{i=1}^{I} f_i(x_i) + \lambda \left[M - \sum_{i=1}^{I} c_i(x_i) \right] + \mu \left[N - \sum_{i=1}^{I} d_i(x_i) \right] \right.$$

Subject to

$$\left. \begin{aligned} & g_i(x_i) = A_i \quad \forall i \\ & h_i(x_i) \le B_i \quad \forall i \end{aligned} \right\}$$

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Is this equivalent problem decomposed?

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Is this equivalent problem decomposed? Not yet!

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Is this equivalent problem decomposed? Not yet!

- Let's **relax** the equivalent problem above by **fixing dual variables** (**λ and μ**) to given values, i.e., $\bar{\lambda}$ **and** $\bar{\mu}$.

LR: Mathematical procedure

The relaxed problem:

$$\underset{x_i}{\text{Minimize}} \sum_{i=1}^{I} f_i(x_i) + \bar{\boldsymbol{\lambda}}\left[M - \sum_{i=1}^{I} c_i(x_i)\right] + \bar{\boldsymbol{\mu}}\left[N - \sum_{i=1}^{I} d_i(x_i)\right]$$

Subject to

$$g_i(x_i) = A_i \qquad \forall i$$

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Subject to

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Is this relaxed problem decomposable?

LR: Mathematical procedure

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$$\underset{x_i}{\text{Minimize}} \sum_{i=1}^{I} f_i(x_i) + \bar{\lambda}\left[M - \sum_{i=1}^{I} c_i(x_i)\right] + \bar{\mu}\left[N - \sum_{i=1}^{I} d_i(x_i)\right]$$

Subject to

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Parameters (fixed values)

Is this relaxed problem decomposable? Yes, one per i:

$$\left\{\underset{x_i}{\text{Minimize}} \; f_i(x_i) + \bar{\lambda}\, c_i(x_i) + \bar{\mu}\, d_i(x_i)\right.$$

Subject to

$$\left.\begin{array}{l} g_i(x_i) = A_i \\ h_i(x_i) \leq B_i \end{array}\right\} \forall i$$

LR: Mathematical procedure

LR is an iterative approach with a systematic way to update the values of fixed dual variables ($\bar{\lambda}$ and $\bar{\mu}$) in each iteration.

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Available techniques in the literature to update $\bar{\lambda}$ and $\bar{\mu}$ [1]:

1. Subgradient method
2. Cutting plane method
3. Bundle method
4. Trust region method
5. …

[1] N. J. Redondo and A. J. Conejo, "Short-term hydro-thermal coordination by Lagrangian relaxation: solution of the dual problem," *IEEE Transactions on Power Systems*, vol. 14, no. 1, pp. 89–95, Feb. 1999.

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} Will <u>not</u> be covered in this course!

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LR: illustrative example

$$\underset{x\geq 0, y\geq 0}{\text{Minimize}} \; x^2 + y^2$$

$$\text{Subject to} \quad -x - y = -4 \qquad (\mu)$$

Note: Objective function includes quadratic terms, so LR works!

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Subproblem 1:

$$\underset{x\geq 0}{\text{Minimize}} \ x^2 - \bar{\mu}x$$

Subproblem 2:

$$\underset{y\geq 0}{\text{Minimize}} \ y^2 - \bar{\mu}y$$

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Subproblem 2:

$$\underset{y\geq 0}{\text{Minimize}} \; y^2 - \bar{\mu}y$$

Updating fixed dual variable ($\bar{\mu}$) using <u>subgradient</u> method:

- Solve subproblems 1 and 2 in iteration v, and obtain the values $x^{(v)}$ and $y^{(v)}$
- $\bar{\mu}^{(v+1)} \leftarrow \bar{\mu}^{(v)} + \frac{1}{a+bv} \frac{-x^{(v)}-y^{(v)}+4}{\left|-x^{(v)}-y^{(v)}+4\right|}$
- a and b are positive constants, e.g., $a = 1$ and $b = 0.1$.

LR: illustrative example

Algorithm:

- **Step 0: Initialization**

  Set $\upsilon = 1$ and $\bar{\mu}^{(1)} = \bar{\mu}^{\text{initial}}$

- **Step 1: Solve subproblems 1 and 2, and obtain $\boldsymbol{x}^{(\boldsymbol{\upsilon})}$ and $\boldsymbol{y}^{(\boldsymbol{\upsilon})}$**

- **Step 2: Update fixed dual variable, i.e., $\overline{\boldsymbol{\mu}}^{(\boldsymbol{\upsilon}+\mathbf{1})}$**

- **Step 3: Convergence check**

  If $\frac{\|\overline{\mu}^{(\upsilon+1)}-\overline{\mu}^{(\upsilon)}\|}{\|\overline{\mu}^{(\upsilon)}\|} \leq \epsilon$, then the optimal solution with a level of accuracy ϵ is obtained, otherwise $\upsilon \leftarrow \upsilon + 1$ and go Step 1

Optional Assignment 1

Provide a GAMS (or Python or Julia) code for the previous illustrative example solved by LR algorithm!

Augmented Lagrangian relaxation (ALR)

Recall:

ALR works for problems with either quadratic objective function (like LR) or linear one (unlike LR)

Main difference of ALR with respect to LR:

An additional penalty term within the subproblems

Augmented Lagrangian relaxation (ALR)

Recall the previous illustrative example:

$$\underset{x\geq 0, y\geq 0}{\text{Minimize}} \; x^2 + y^2$$

Subject to $-x - y = -4 \quad (\lambda)$

Augmented Lagrangian relaxation (ALR)

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Equivalent to:

$$\underset{\lambda}{\text{Maximize}} \; \underset{x \geq 0, y \geq 0}{\text{Minimize}} \; x^2 + y^2 + \lambda(-x - y + 4)$$

Augmented Lagrangian relaxation (ALR)

Recall the previous illustrative example:

$$\underset{x\geq 0, y\geq 0}{\text{Minimize}}\ x^2 + y^2$$

Subject to $-x - y = -4 \quad (\lambda)$

Additional penalty term with respect to LR, whose value is zero in the optimal point.
γ is a positive constant

Equivalent to:

$$\underset{\lambda}{\text{Maximize}}\ \underset{x\geq 0, y\geq 0}{\text{Minimize}}\ x^2 + y^2 + \lambda(-x - y + 4) + \frac{\gamma}{2}\|-x - y + 4\|^2$$

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Since it is a quadratic term, the first derivatives of the objective function with respect to variables are now continuous (not fixed). This is necessary for convergence.

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Question:

- **Similar to LR, assume dual variable λ is fixed to a given value $\bar{\lambda}$. Is the problem above decomposed for given $\bar{\lambda}$?**

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Question:

- **Similar to LR, assume dual variable λ is fixed to a given value $\bar{\lambda}$. Is the problem above decomposed for given $\bar{\lambda}$? No, due to product of x and y in the penalty term!**

Augmented Lagrangian relaxation (ALR)

Available alternatives to solve ALR:

- Auxiliary problem principle (APP)
- Alternating direction method of multipliers (ADMM)

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❑ ADMM directly fixes each variable to its value obtained in the previous iteration, and decomposes the ALR to subproblems.

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- Auxiliary problem principle (APP): *will not be covered in this course*
- Alternating direction method of multipliers (ADMM)

❑ ADMM directly fixes each variable to its value obtained in the previous iteration, and decomposes the ALR to subproblems.

❑ Proof for convergence of ADMM to the optimal solution (providing that the original problem is convex) is available in [1].

[1] S. Boyd, N. Parikh, E. Chu, B. Peleato, and J. Eckstein, "Distributed optimization and statistical learning via the alternating direction method of multipliers," *Foundations and Trends in Machine Learning*, vol. 3, no. 1, pp. 1-122, Jan. 2011.

ADMM

The equivalent of original problem:

$$\underset{\lambda}{\text{Maximize}} \ \underset{x\geq 0, y\geq 0}{\text{Minimize}} \ x^2 + y^2 + \lambda(-x-y+4) + \frac{\gamma}{2}\|-x-y+4\|^2$$

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Using ADMM, the problem above in iteration υ can be decomposed to two subproblems:

$$\begin{cases} \underset{x^{(\upsilon)}\geq 0}{\text{Minimize}}\ x^{2(\upsilon)} - \lambda^{(\upsilon-1)}x^{(\upsilon)} + \dfrac{\gamma}{2}\left\|-x^{(\upsilon)} - y^{(\upsilon-1)} + 4\right\|^2 \\ \underset{y^{(\upsilon)}\geq 0}{\text{Minimize}}\ y^{2(\upsilon)} - \lambda^{(\upsilon-1)}y^{(\upsilon)} + \dfrac{\gamma}{2}\left\|-y^{(\upsilon)} - x^{(\upsilon-1)} + 4\right\|^2 \end{cases}$$

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In the first subproblem above (i.e., x-update), $x^{(\upsilon)}$ is variable, while $\lambda^{(\upsilon-1)}$ and $y^{(\upsilon-1)}$ are parameters.

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In the second subproblem above (i.e., y-update), $y^{(\upsilon)}$ is variable, while $\lambda^{(\upsilon-1)}$ and $x^{(\upsilon-1)}$ are parameters.

ADMM

The equivalent of original problem:

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λ-update: $\lambda^{(\upsilon)} \leftarrow \lambda^{(\upsilon-1)} + \gamma\left(-x^{(\upsilon)} - y^{(\upsilon)} + 4\right)$

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λ-update: $\lambda^{(\upsilon)} \leftarrow \lambda^{(\upsilon-1)} + \gamma\left(-x^{(\upsilon)} - y^{(\upsilon)} + 4\right)$

Algorithm: in each iteration, solve each subproblem and then update dual variable untill convergnce, i.e., when the primal residual (i.e., the value of penalty) is negligable, and therefore the value of dual variable does not change anymore.

Optional Assignment 2

Provide a GAMS (or Python or Julia) code for the previous illustrative example solved by ADMM algorithm!

Thanks for your attention!

Email: seykaz@elektro.dtu.dk