DTU CEE Summer School 2019

Data-Driven Analytics and Optimization for Energy Systems

June 16-21 2019

Distributed Optimization

Lecture 2: Benders' decomposition

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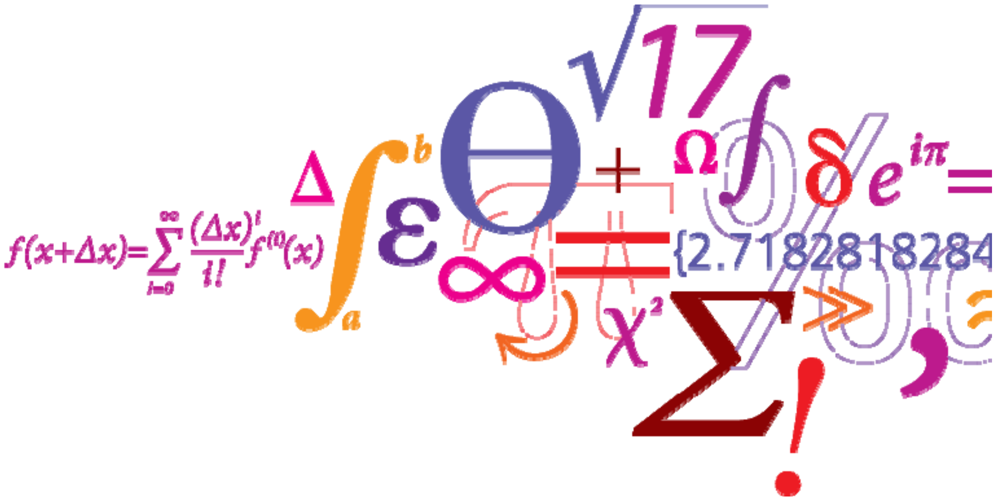

June 21, 2019

DTU Electrical Engineering
Department of Electrical Engineering

Learning objectives

After Lecture 2, you are expected to be able to:

- Explain the functioning of Benders' decomposition
- Implement it to an illustrative problem

Recall from previous lecture

Single-node single-year (static) generation expansion problem, considering two existing generators (units 1 and 2) and one candidate generator (unit 3)

$$\underset{g_{h,1},g_{h,2},g_{h,3},x_3}{\text{Minimize}} \quad 15000x_3 + \sum_h \left[10g_{h,1} + 25g_{h,2} + 30g_{h,3}\right]$$

subject to

$$0 \leq g_{h,1} \leq 100 \quad \forall h$$

$$0 \leq g_{h,2} \leq 150 \quad \forall h$$

$$0 \leq g_{h,3} \leq x_3 \quad \forall h$$

$$g_{h,1} + g_{h,2} + g_{h,3} = D_{h,1} \quad \forall h$$

$$x_3 \geq 0$$

Recall from previous lecture

Single-node single-year (static) generation expansion problem, considering two existing generators (units 1 and 2) and one candidate generator (unit 3)

$$\underset{g_{h,1}, g_{h,2}, g_{h,3}, x_3}{\text{Minimize}} \quad \underbrace{15000 x_3}_{\text{Expansion cost of candidate unit}} + \underbrace{\sum_h [10 g_{h,1} + 25 g_{h,2} + 30 g_{h,3}]}_{\text{Operational cost of (existing and candidate) units}}$$

subject to

$$0 \leq g_{h,1} \leq 100 \quad \forall h$$

$$0 \leq g_{h,2} \leq 150 \quad \forall h$$

$$0 \leq g_{h,3} \leq x_3 \quad \forall h$$

$$g_{h,1} + g_{h,2} + g_{h,3} = D_{h,1} \quad \forall h$$

$$x_3 \geq 0$$

Recall from previous lecture

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$$g_{h,1} + g_{h,2} + g_{h,3} = D_{h,1} \quad \forall h$$

$$x_3 \geq 0$$

By fixing complicating variable x_3:

- Number of complicating variables: ?
- Number of subproblem: ?

Recall from previous lecture

Single-node single-year (static) generation expansion problem, considering two existing generators (units 1 and 2) and one candidate generator (unit 3)

$$\underset{g_{h,1},g_{h,2},g_{h,3},x_3}{\text{Minimize}} \quad 15000x_3 + \sum_h \left[10g_{h,1} + 25g_{h,2} + 30g_{h,3}\right]$$

subject to

$$0 \leq g_{h,1} \leq 100 \quad \forall h$$

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$$0 \leq g_{h,3} \leq x_3 \quad \forall h$$

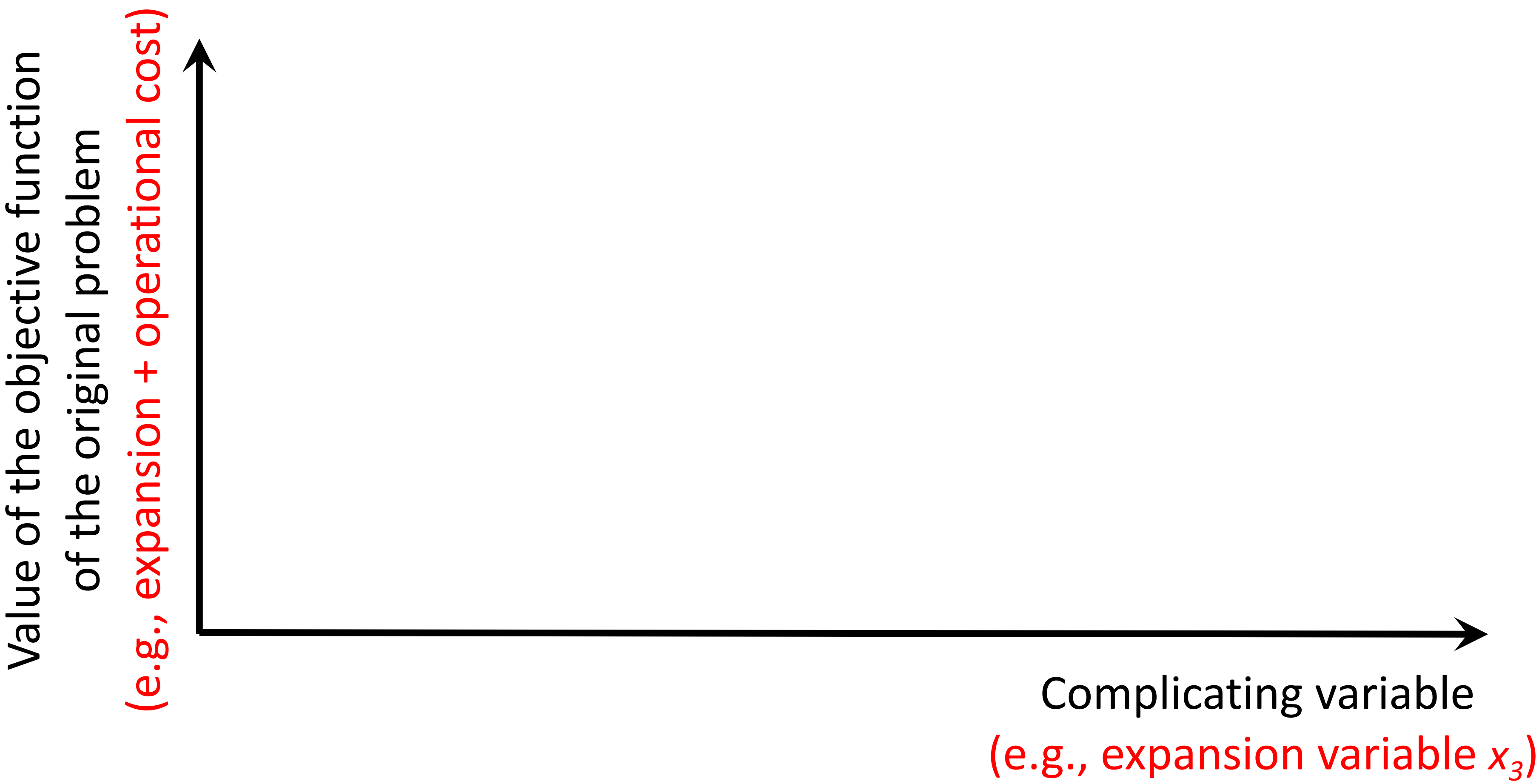
$$g_{h,1} + g_{h,2} + g_{h,3} = D_{h,1} \quad \forall h$$

$$x_3 \geq 0$$

By fixing complicating variable x_3:

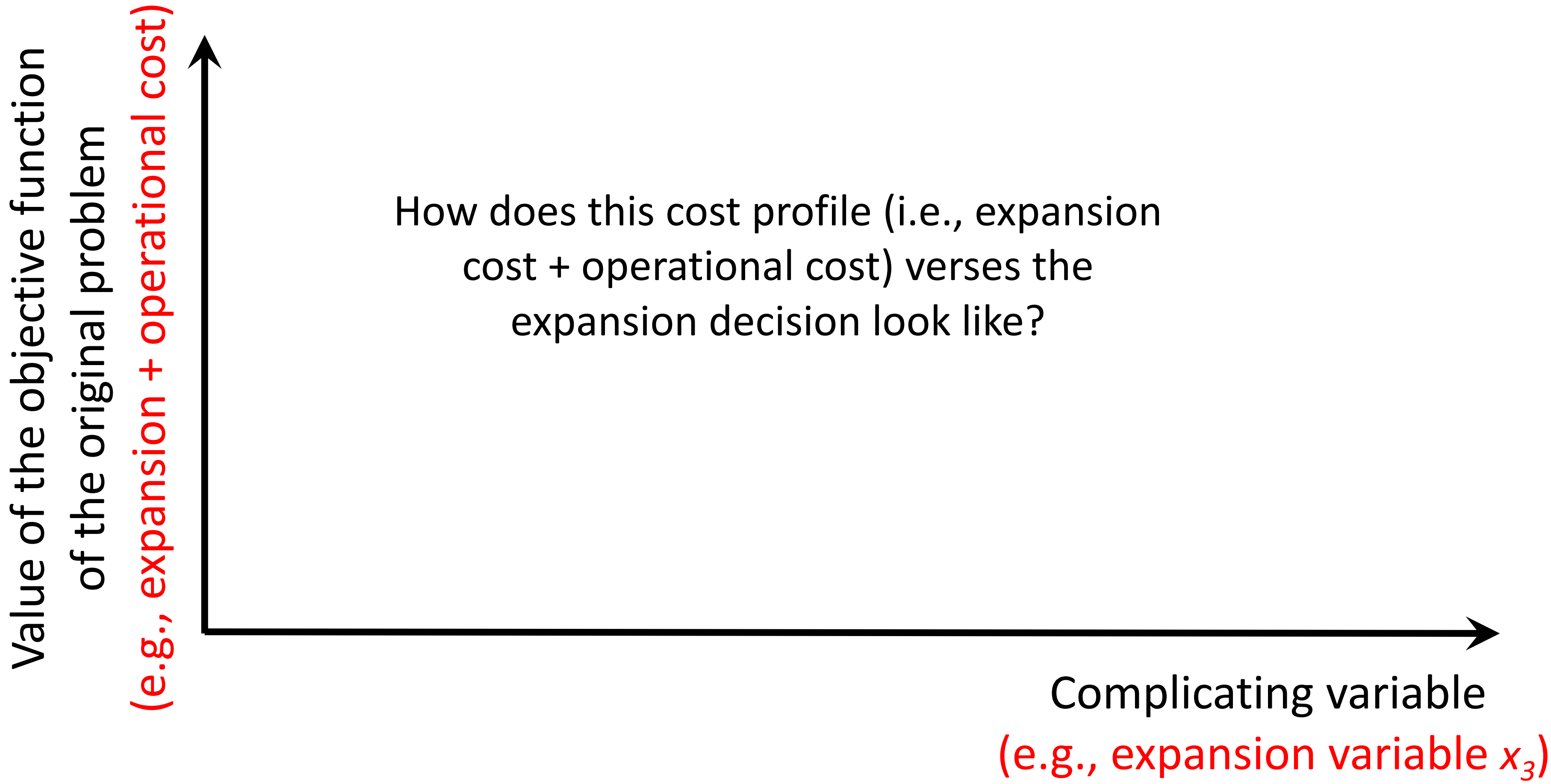
- Number of complicating variables: 1
- Number of subproblem: |h|

Concept

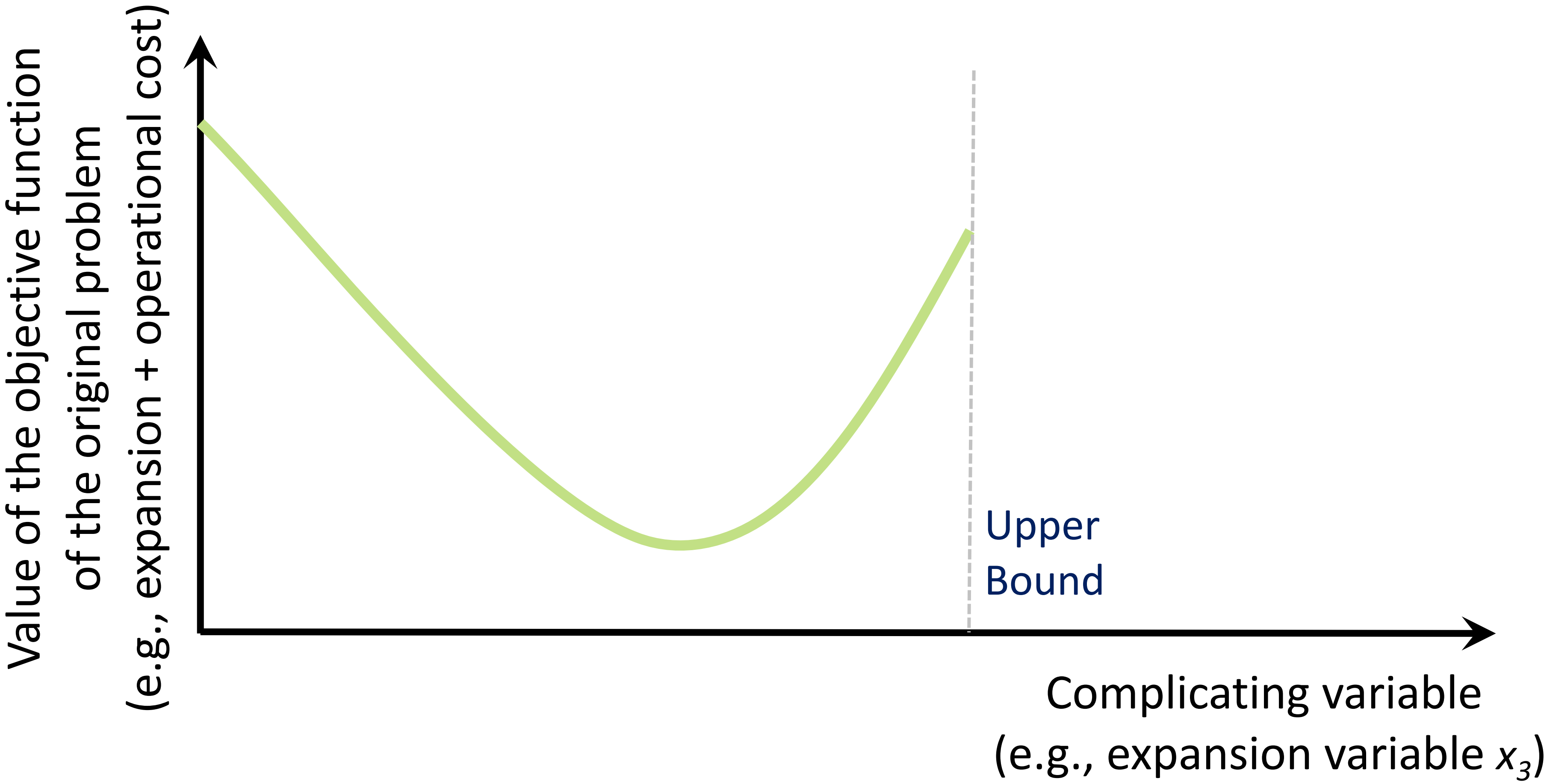

Value of the objective function
of the original problem
(e.g., expansion + operational cost)
Complicating variable
(e.g., expansion variable x_3)

Concept

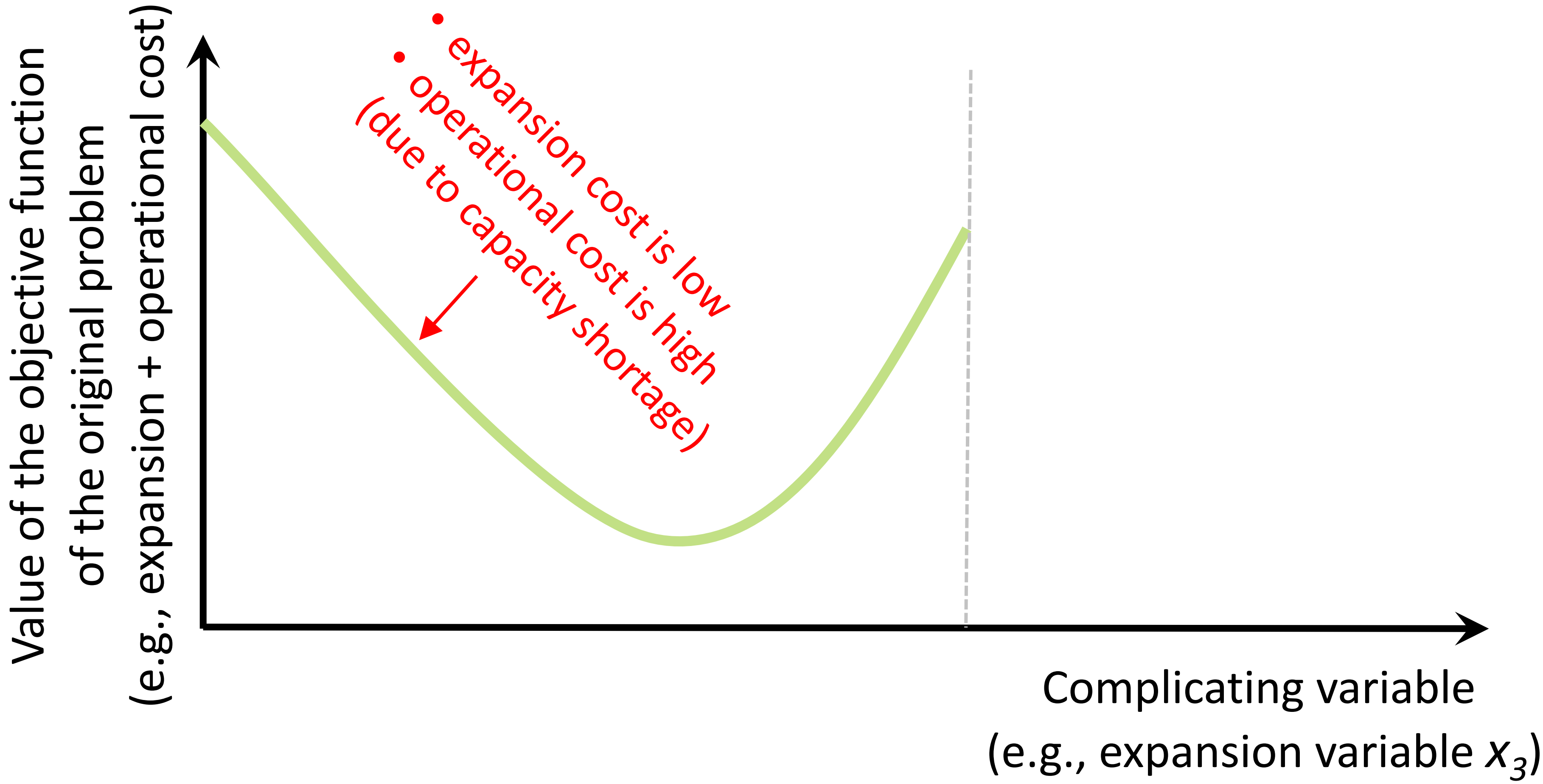
How does this cost profile (i.e., expansion cost + operational cost) verses the expansion decision look like?



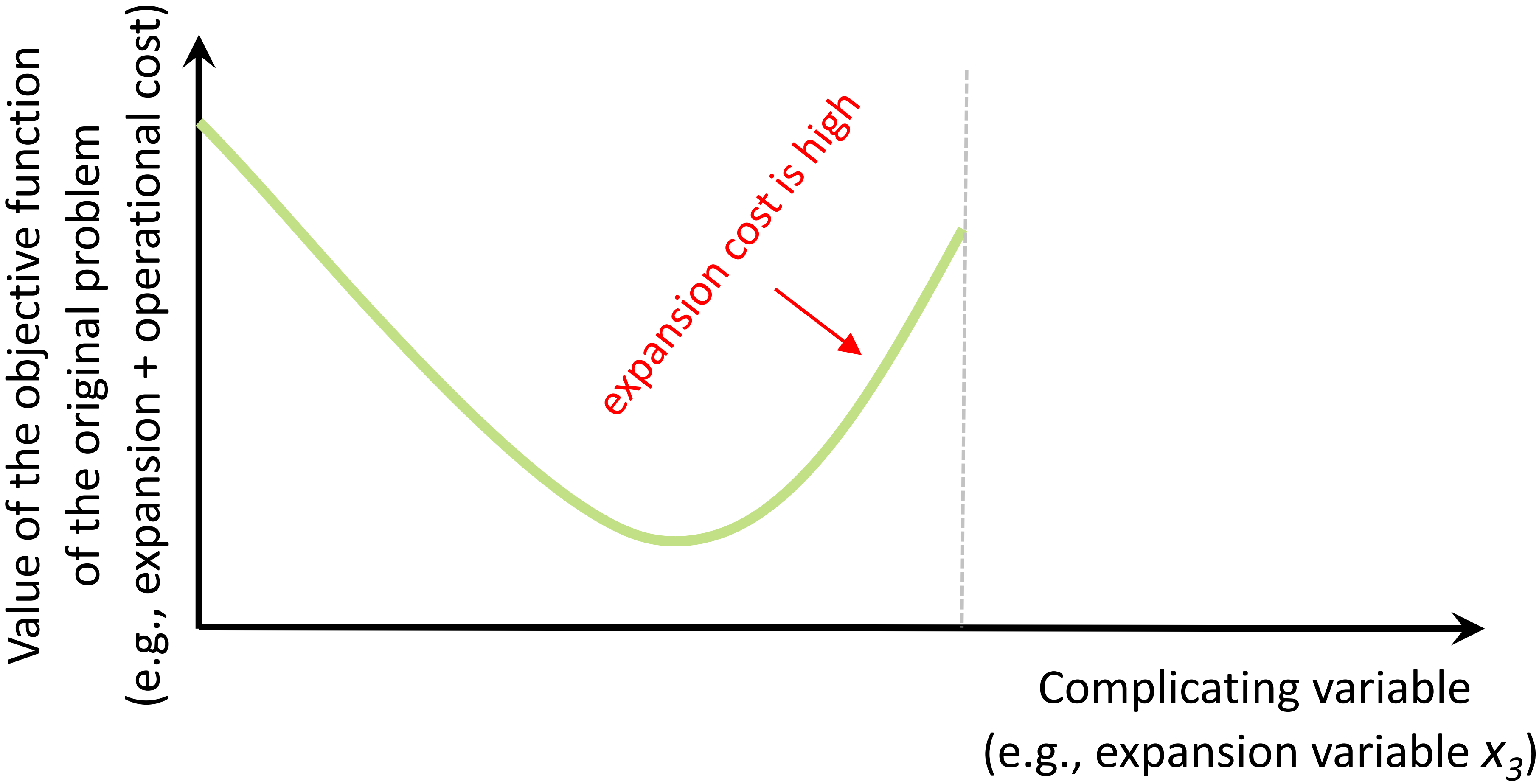
Concept


Value of the objective function
of the original problem
(e.g., expansion + operational cost)
Upper
Bound
Complicating variable
(e.g., expansion variable x_3)

Concept

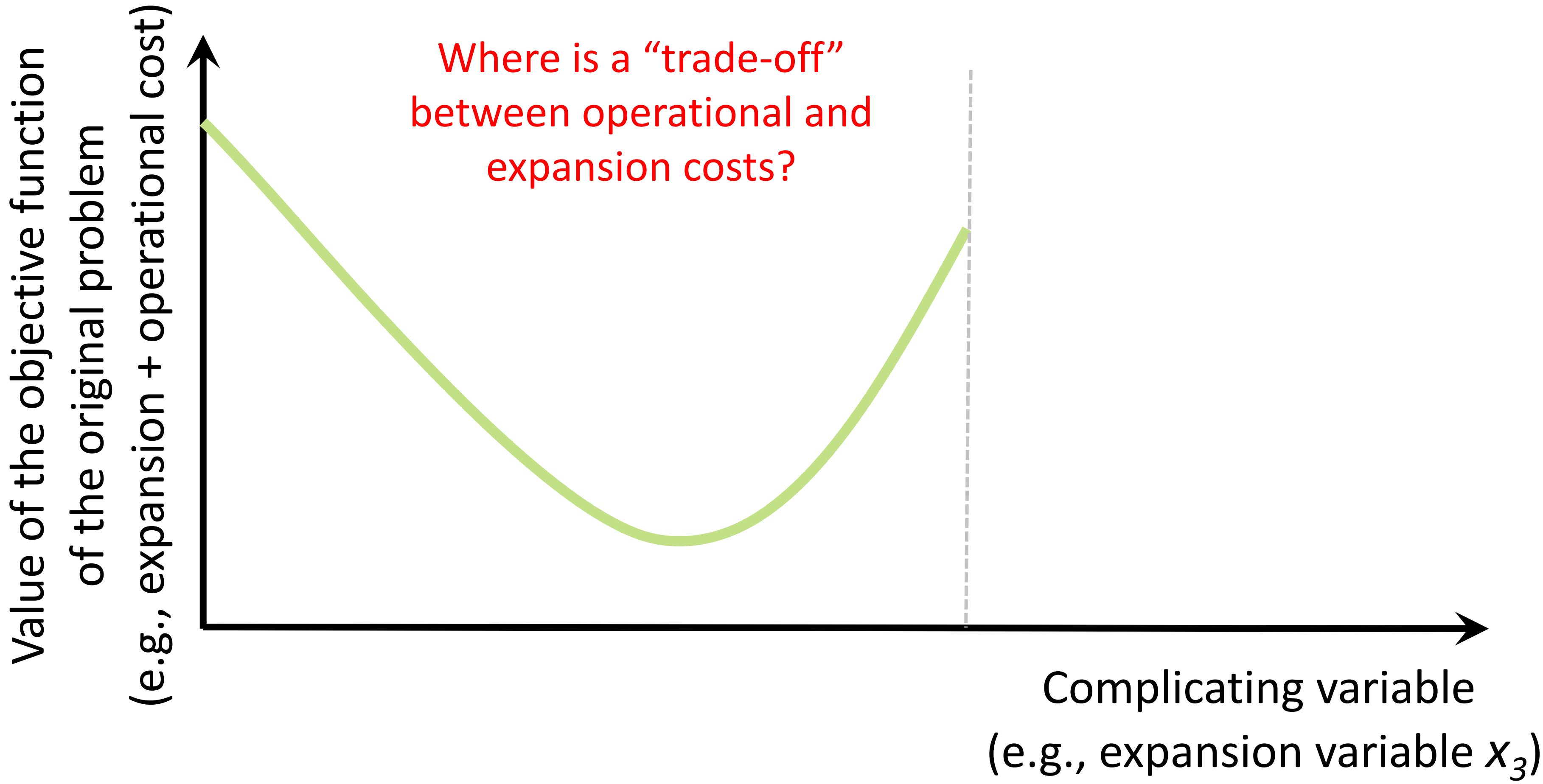

Value of the objective function
of the original problem
(e.g., expansion + operational cost)
• expansion cost is low
• operational cost is high
(due to capacity shortage)
Complicating variable
(e.g., expansion variable x_3)

Concept



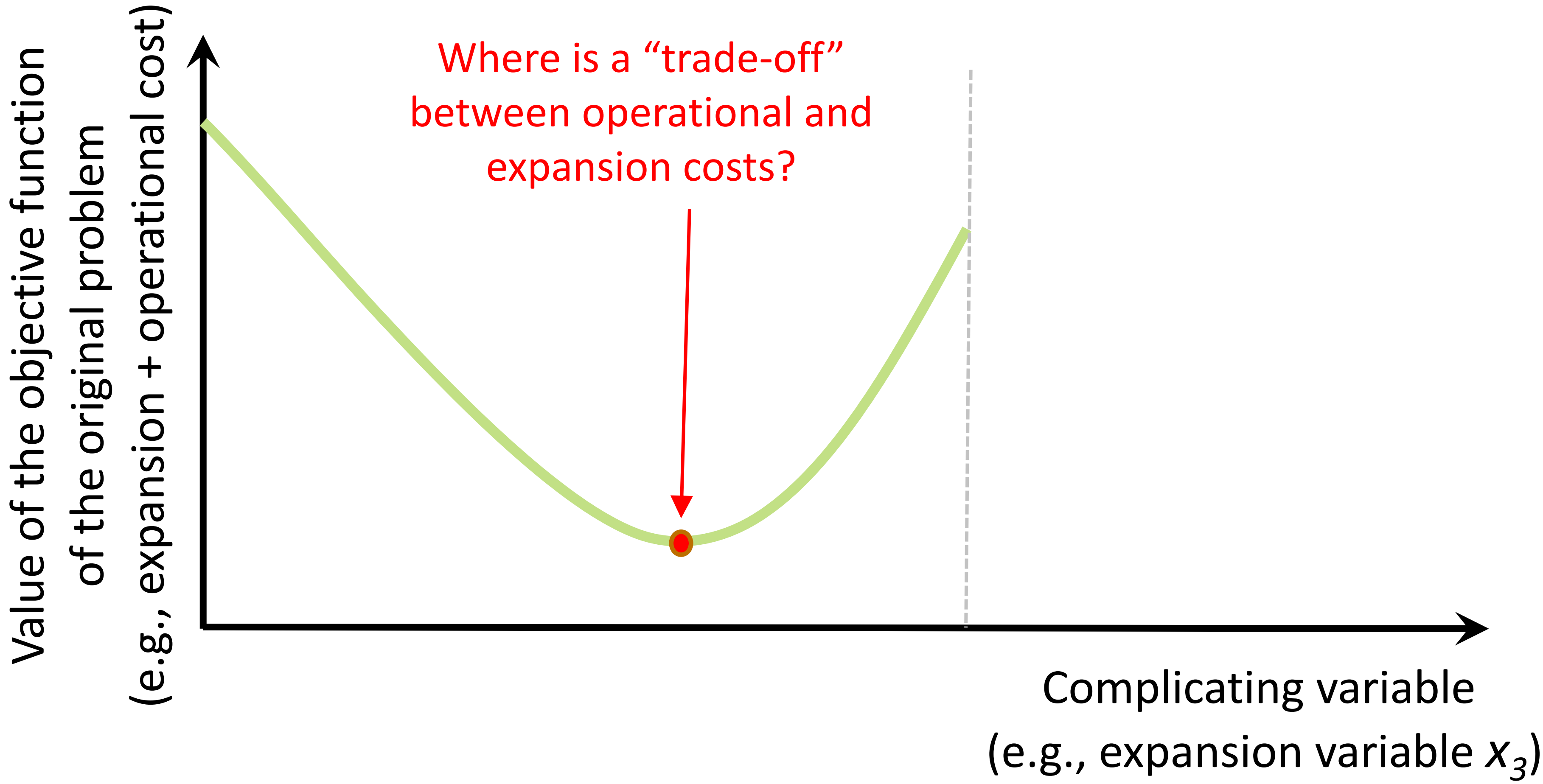
Value of the objective function
of the original problem
(e.g., expansion + operational cost)
expansion cost is high
Complicating variable
(e.g., expansion variable x_3)

Concept

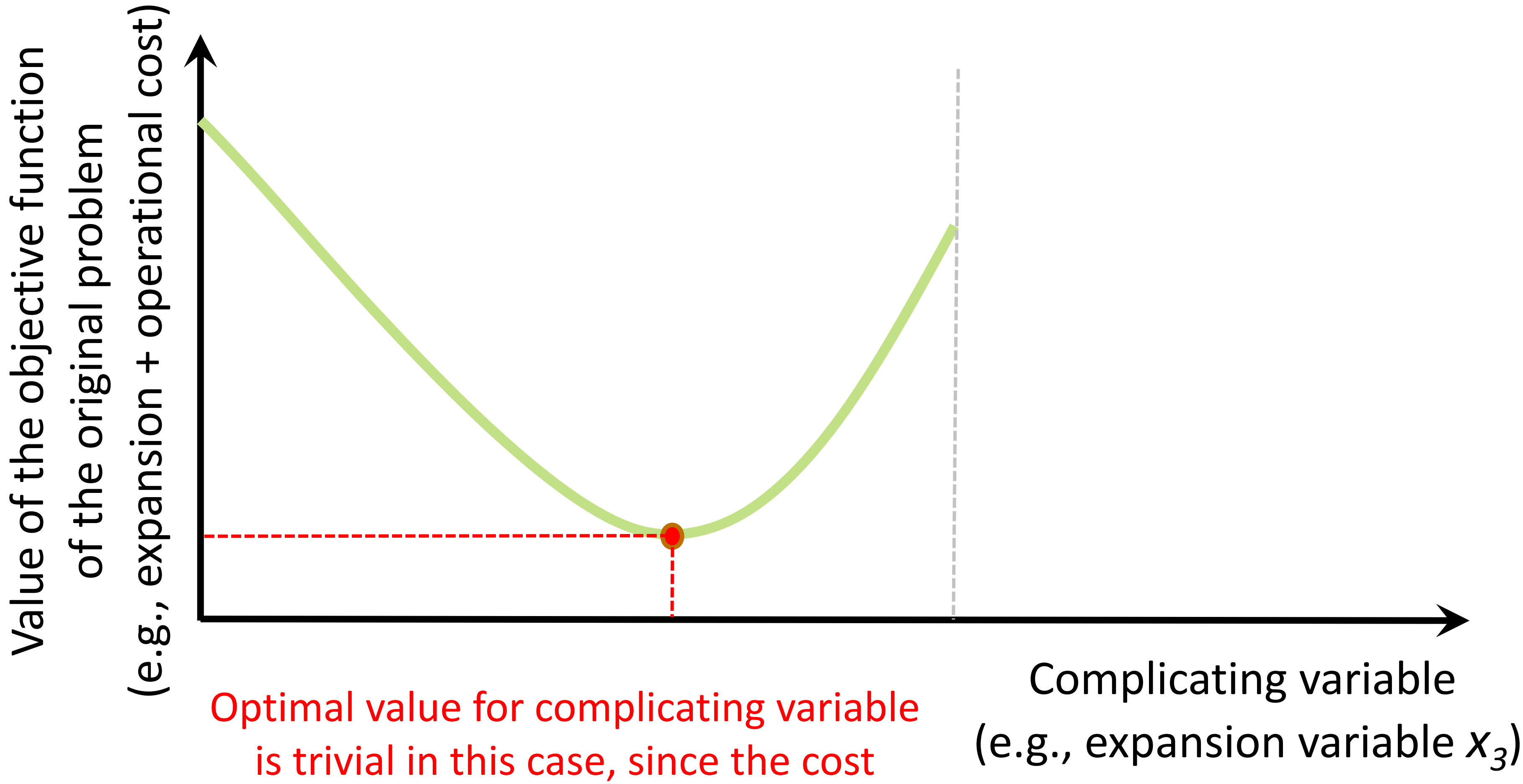


Value of the objective function
of the original problem
(e.g., expansion + operational cost)
Where is a "trade-off" between operational and expansion costs?
Complicating variable
(e.g., expansion variable x_3)

Concept


Value of the objective function
of the original problem
(e.g., expansion + operational cost)
Where is a "trade-off"
between operational and
expansion costs?
Complicating variable
(e.g., expansion variable x_3)

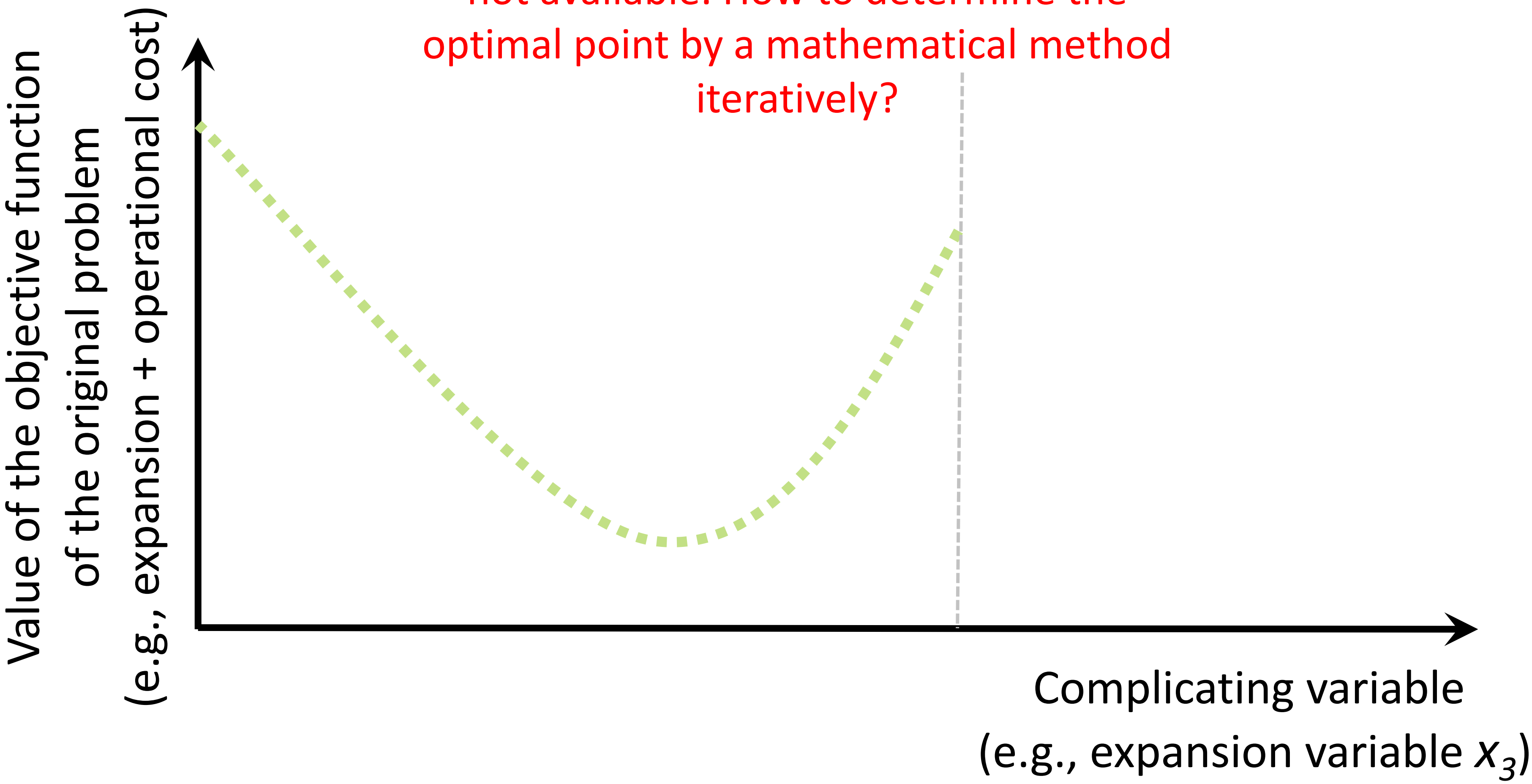
Concept



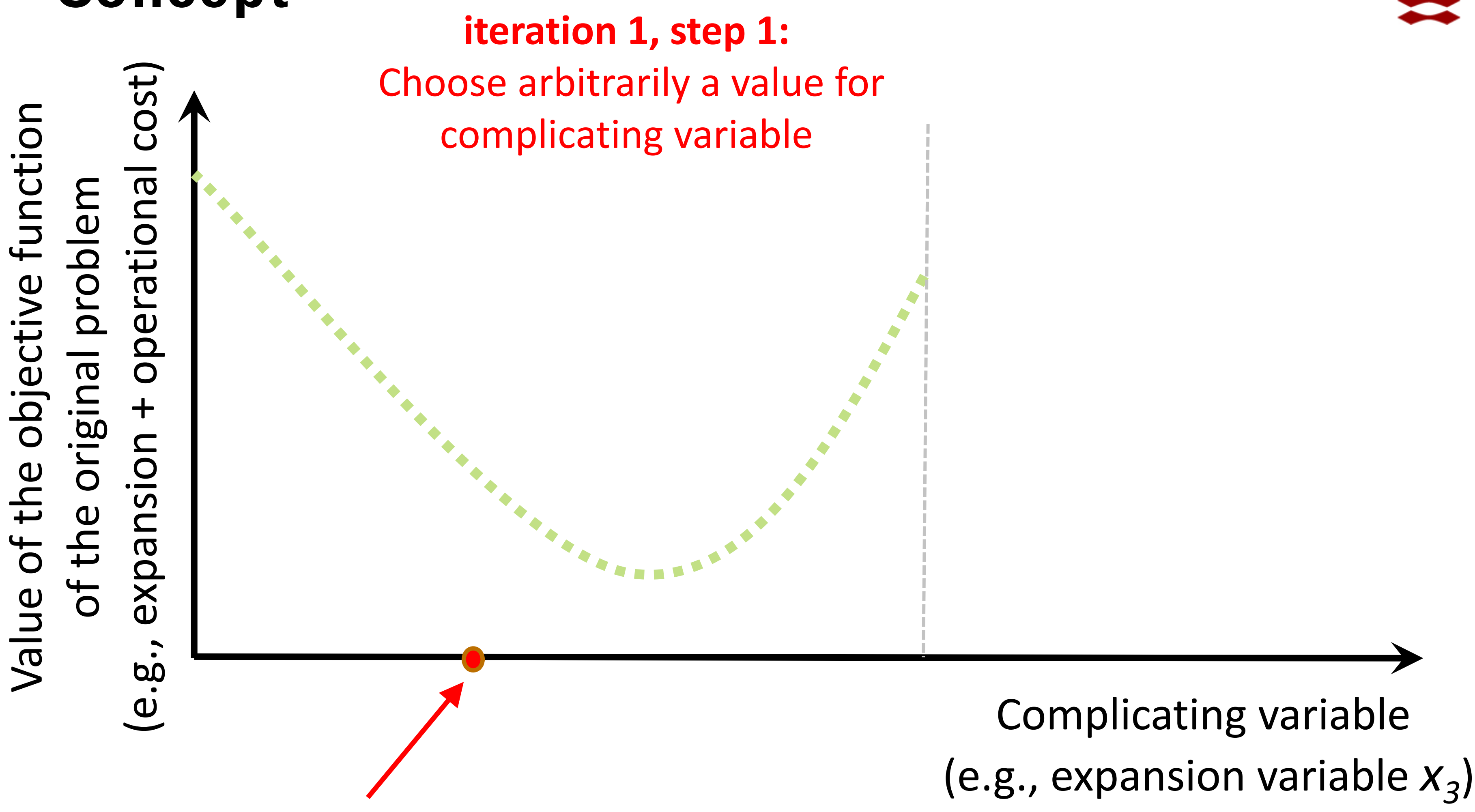
Optimal value for complicating variable is trivial in this case, since the cost profile is available! But what if it is not?

Concept

Assume the cost profile (dotted curve) is not available. How to determine the optimal point by a mathematical method iteratively?

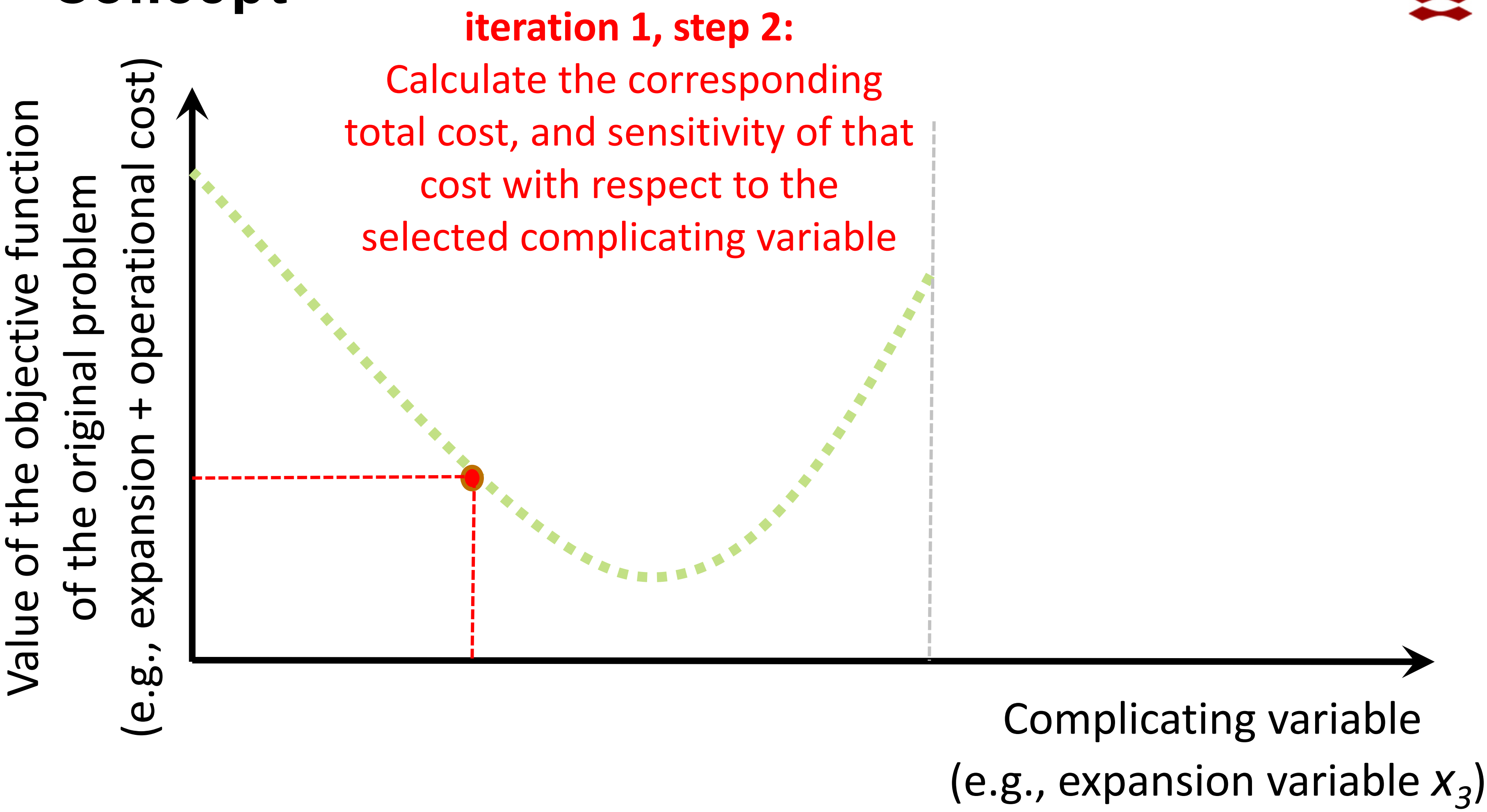


Concept



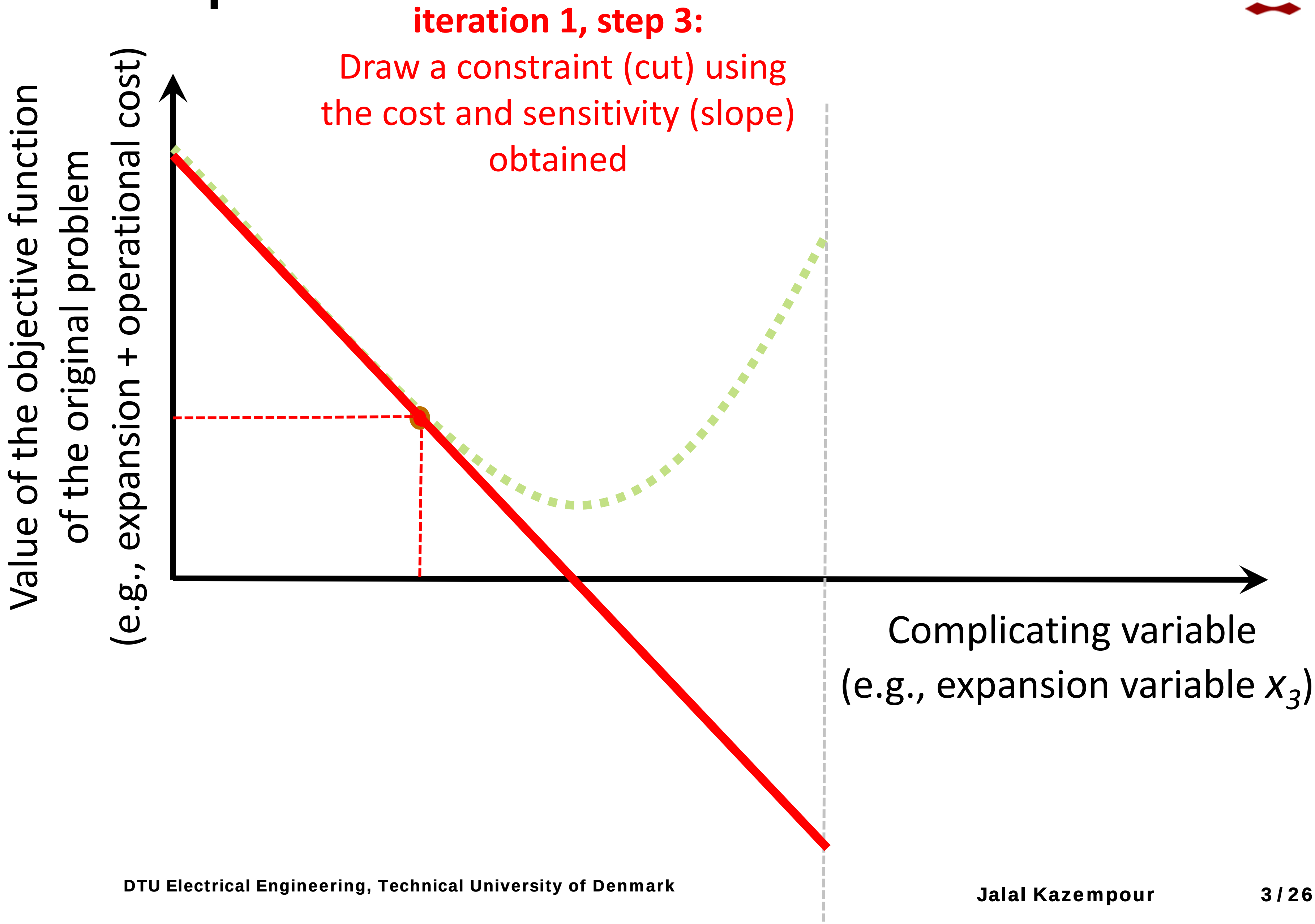
iteration 1, step 1:
Choose arbitrarily a value for complicating variable
Value of the objective function of the original problem (e.g., expansion + operational cost)
Complicating variable (e.g., expansion variable x_3)

Concept



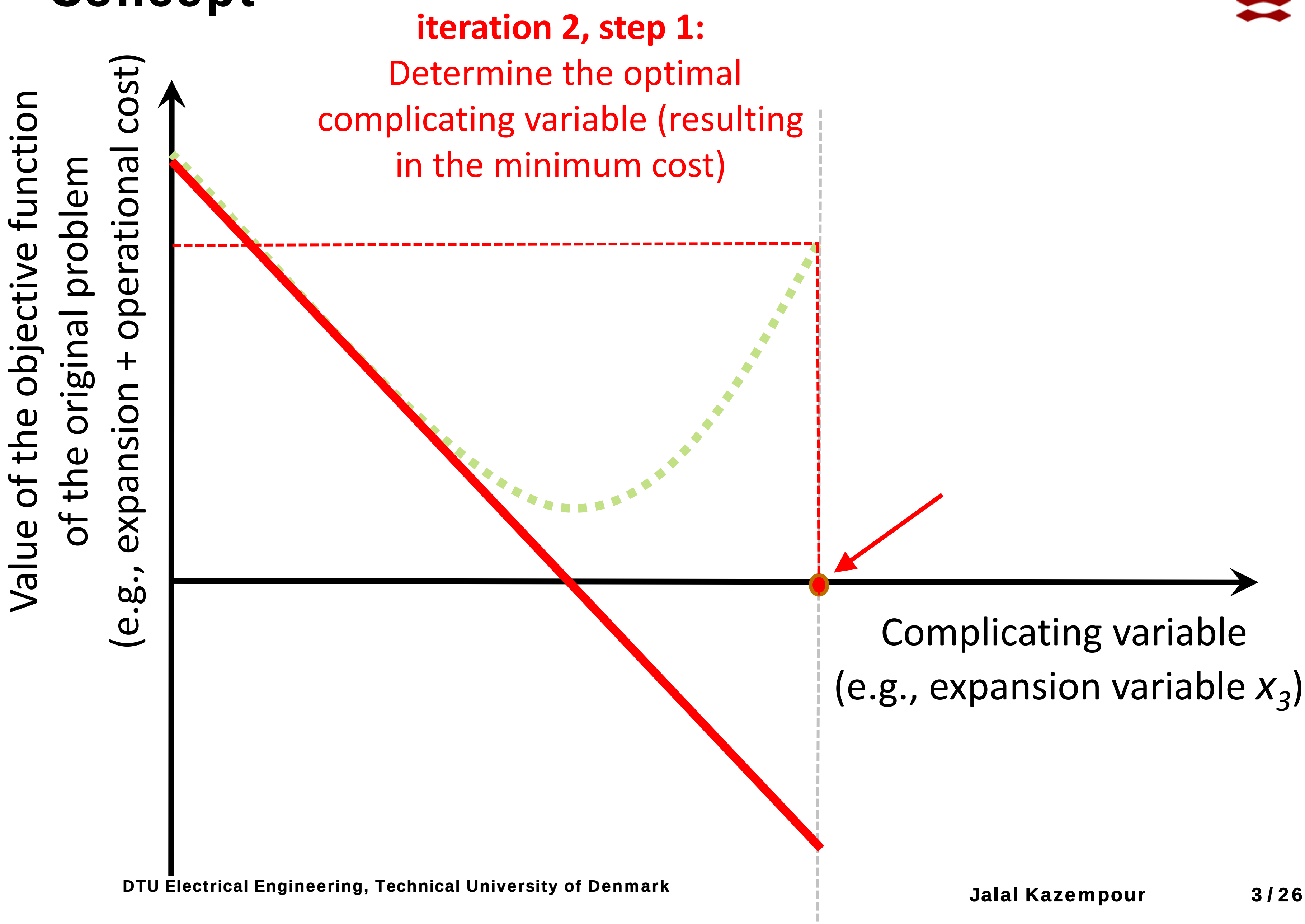
iteration 1, step 2:
Calculate the corresponding total cost, and sensitivity of that cost with respect to the selected complicating variable
Value of the objective function of the original problem (e.g., expansion + operational cost)
Complicating variable (e.g., expansion variable x_3)

Concept



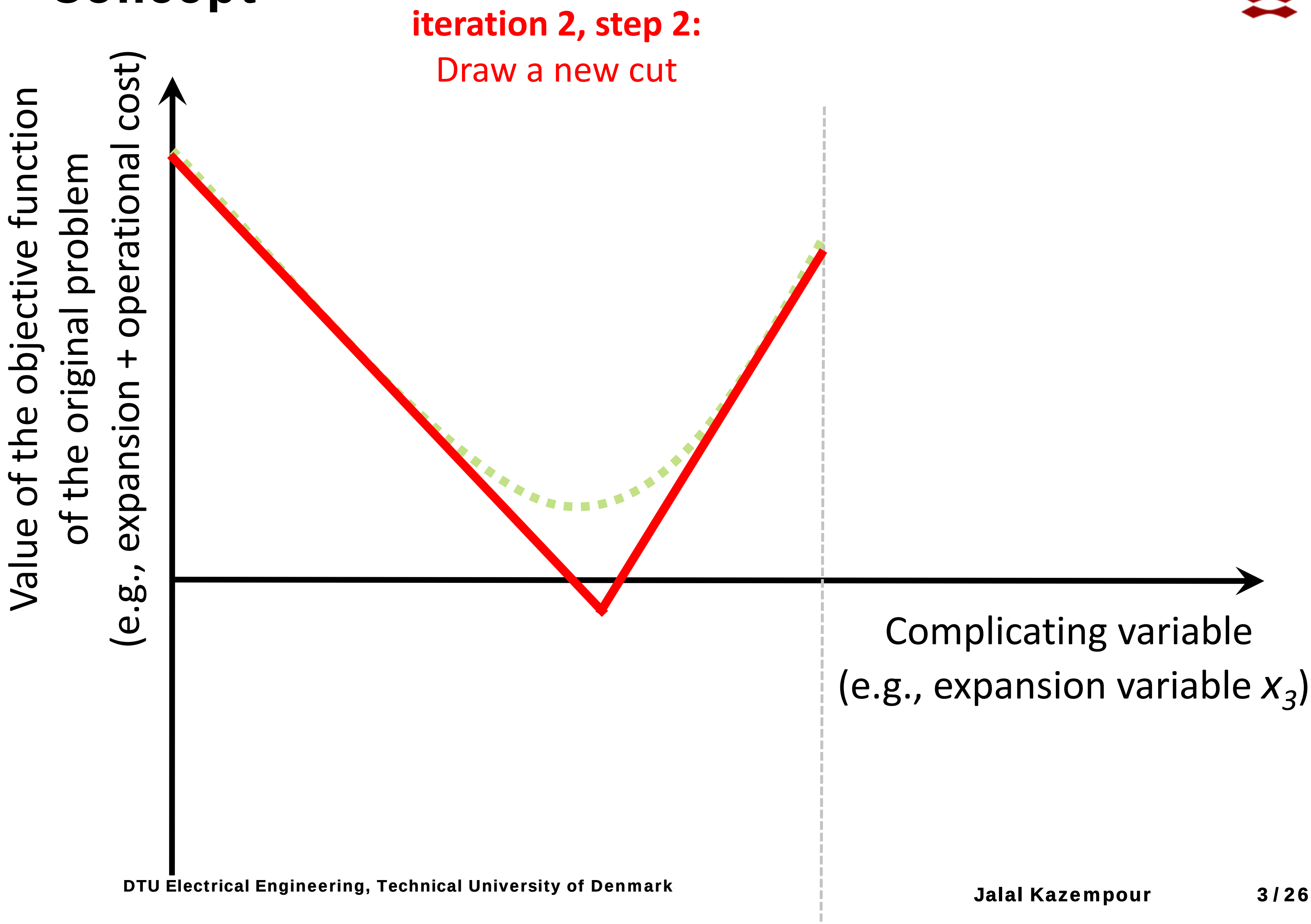
iteration 1, step 3:
Draw a constraint (cut) using the cost and sensitivity (slope) obtained
Value of the objective function of the original problem (e.g., expansion + operational cost)
Complicating variable (e.g., expansion variable x_3)

Concept



iteration 2, step 1:
Determine the optimal complicating variable (resulting in the minimum cost)
Value of the objective function of the original problem (e.g., expansion + operational cost)
Complicating variable (e.g., expansion variable x_3)

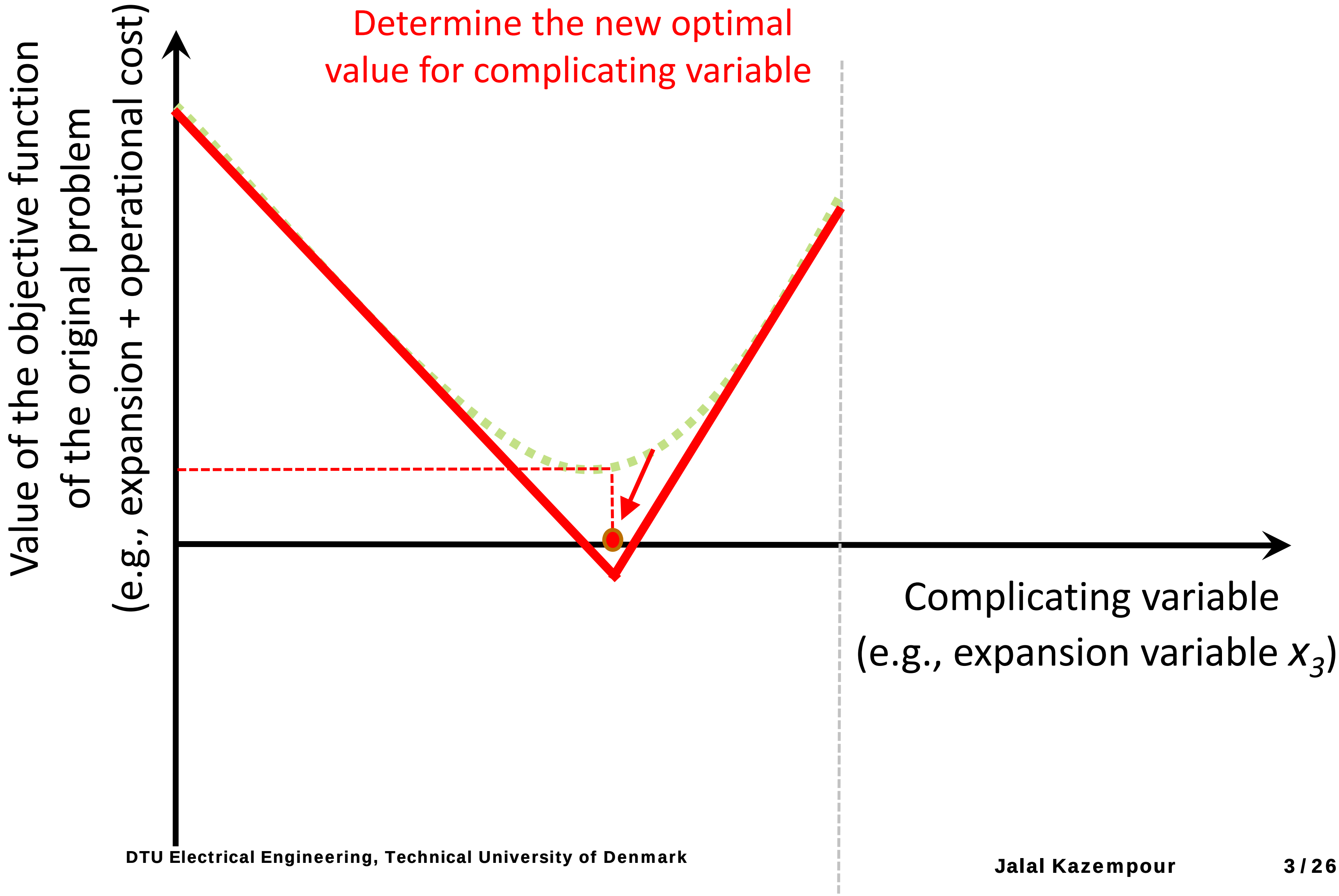
Concept



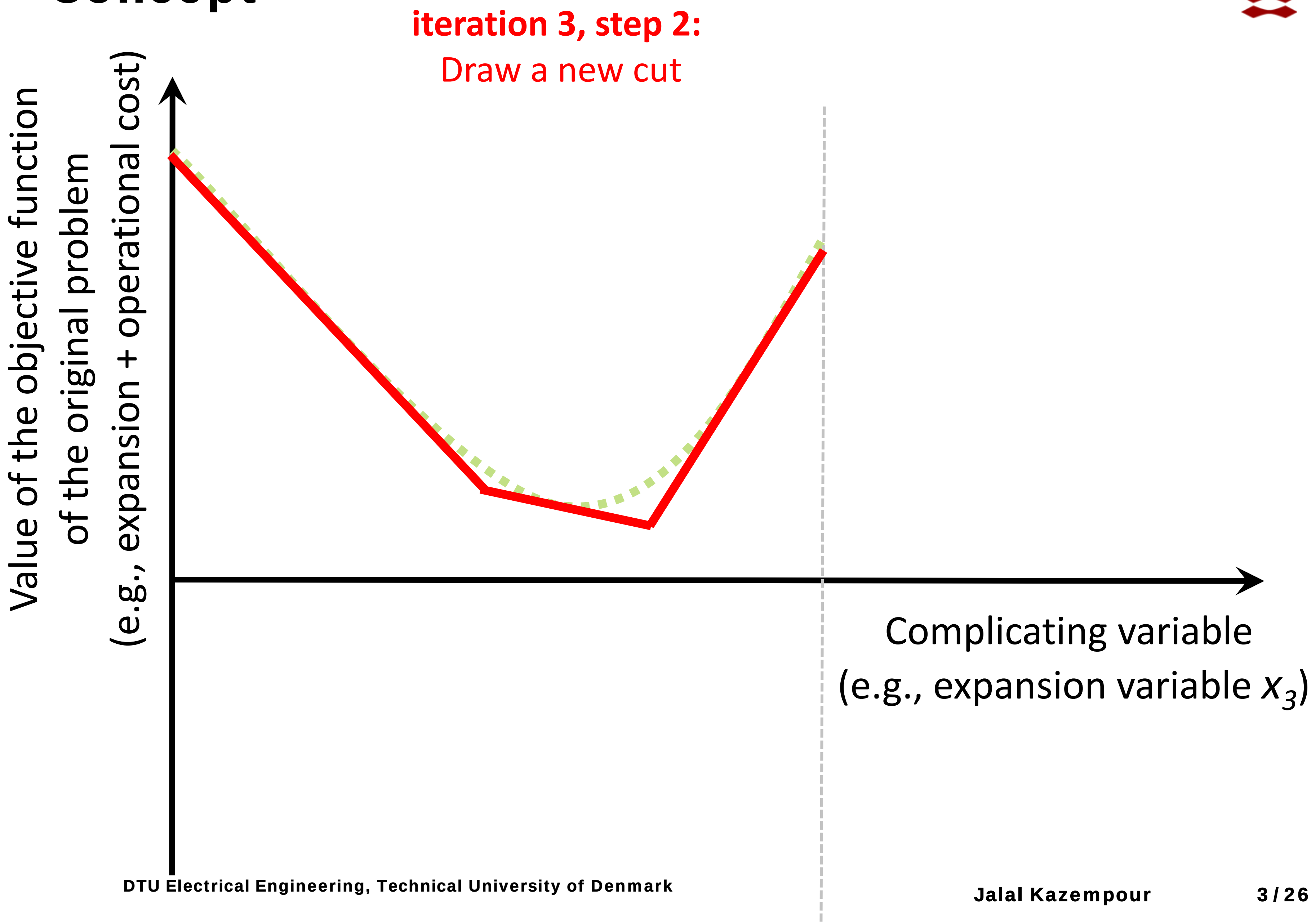
iteration 2, step 2:
Draw a new cut
Value of the objective function
of the original problem
(e.g., expansion + operational cost)
Complicating variable
(e.g., expansion variable x_3)

Concept

iteration 3, step 1:
Determine the new optimal value for complicating variable

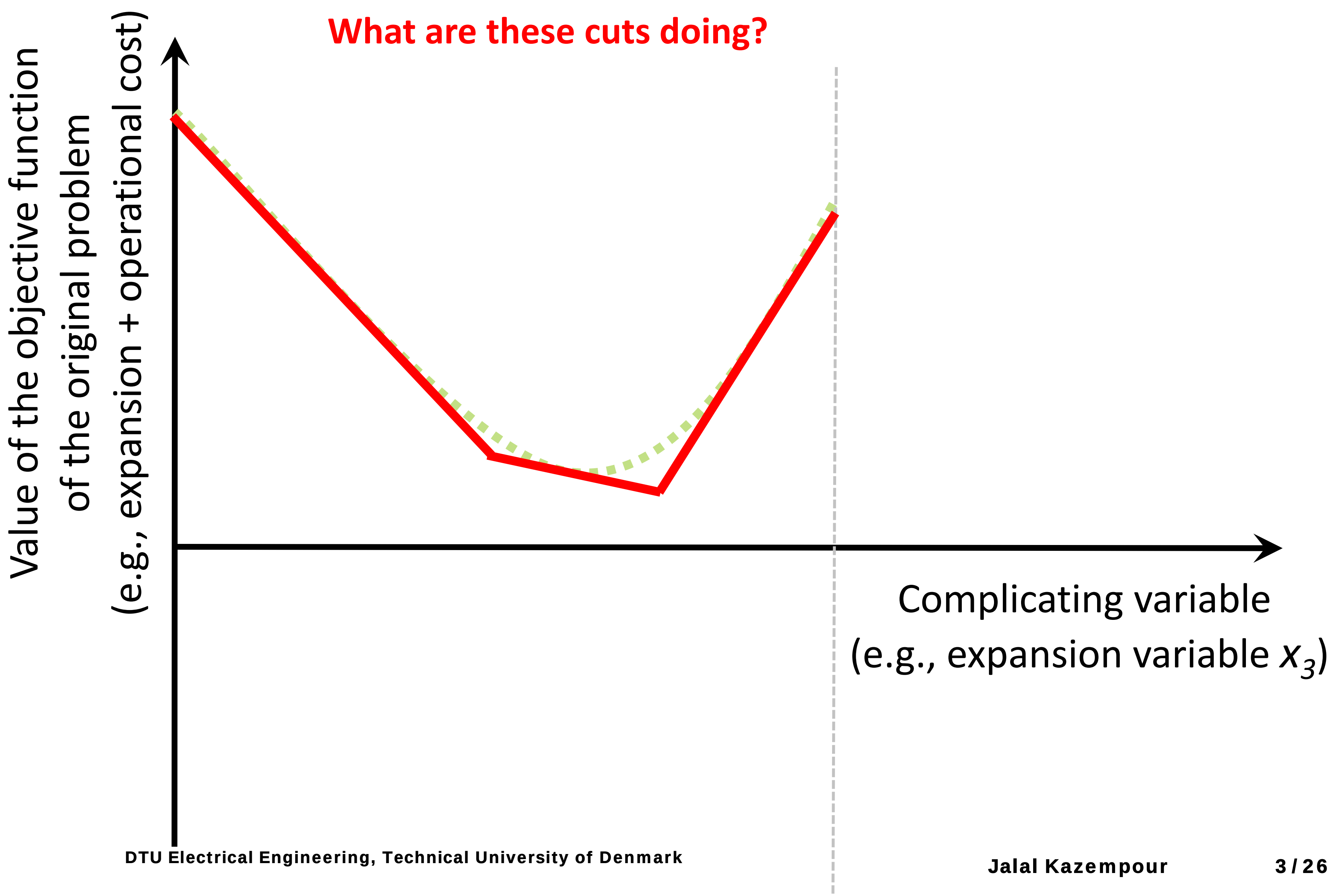


Concept



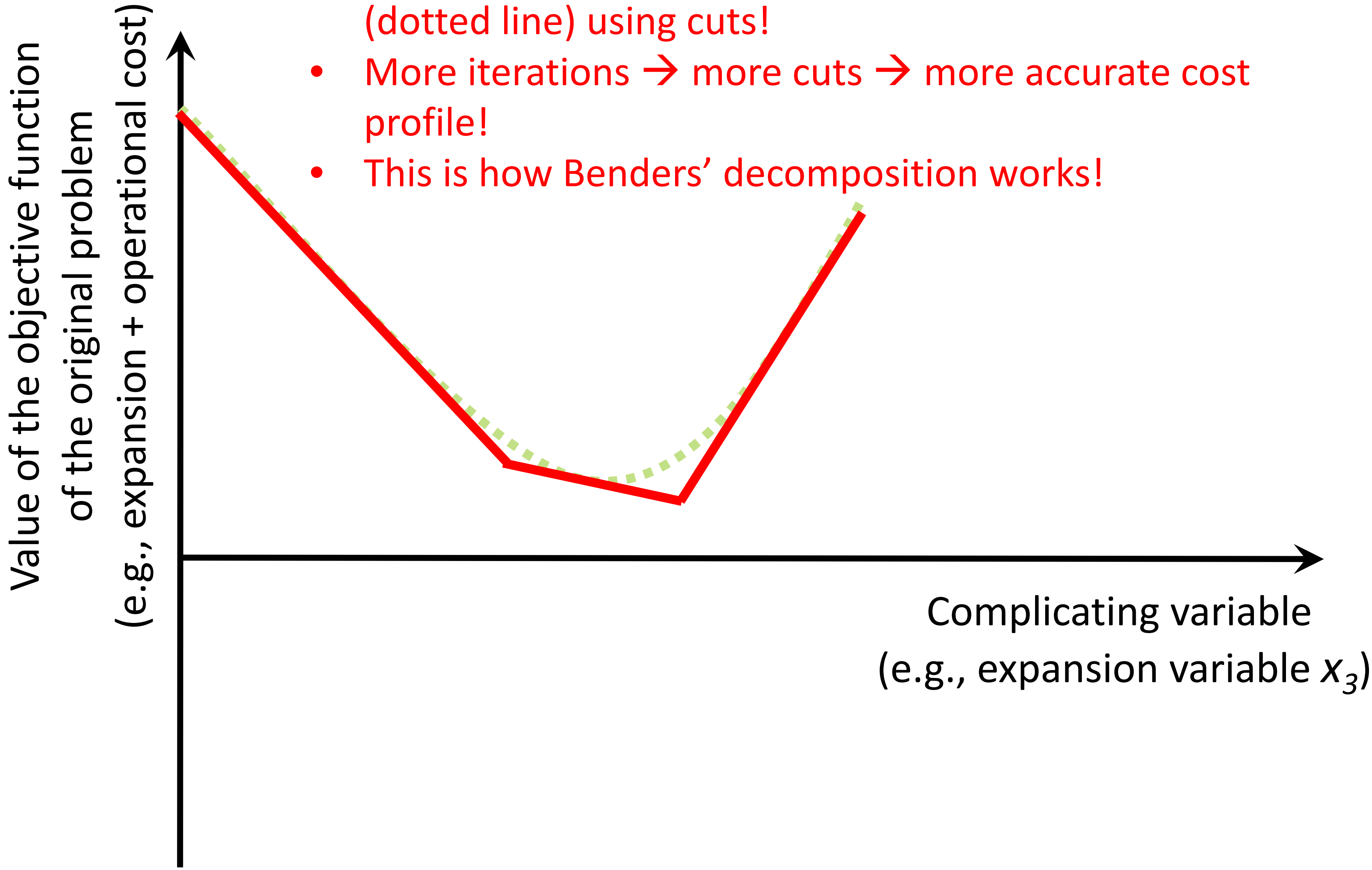
iteration 3, step 2:
Draw a new cut
Value of the objective function of the original problem (e.g., expansion + operational cost)
Complicating variable (e.g., expansion variable x_3)

Concept


What are these cuts doing?
Value of the objective function
of the original problem
(e.g., expansion + operational cost)
Complicating variable
(e.g., expansion variable x_3)

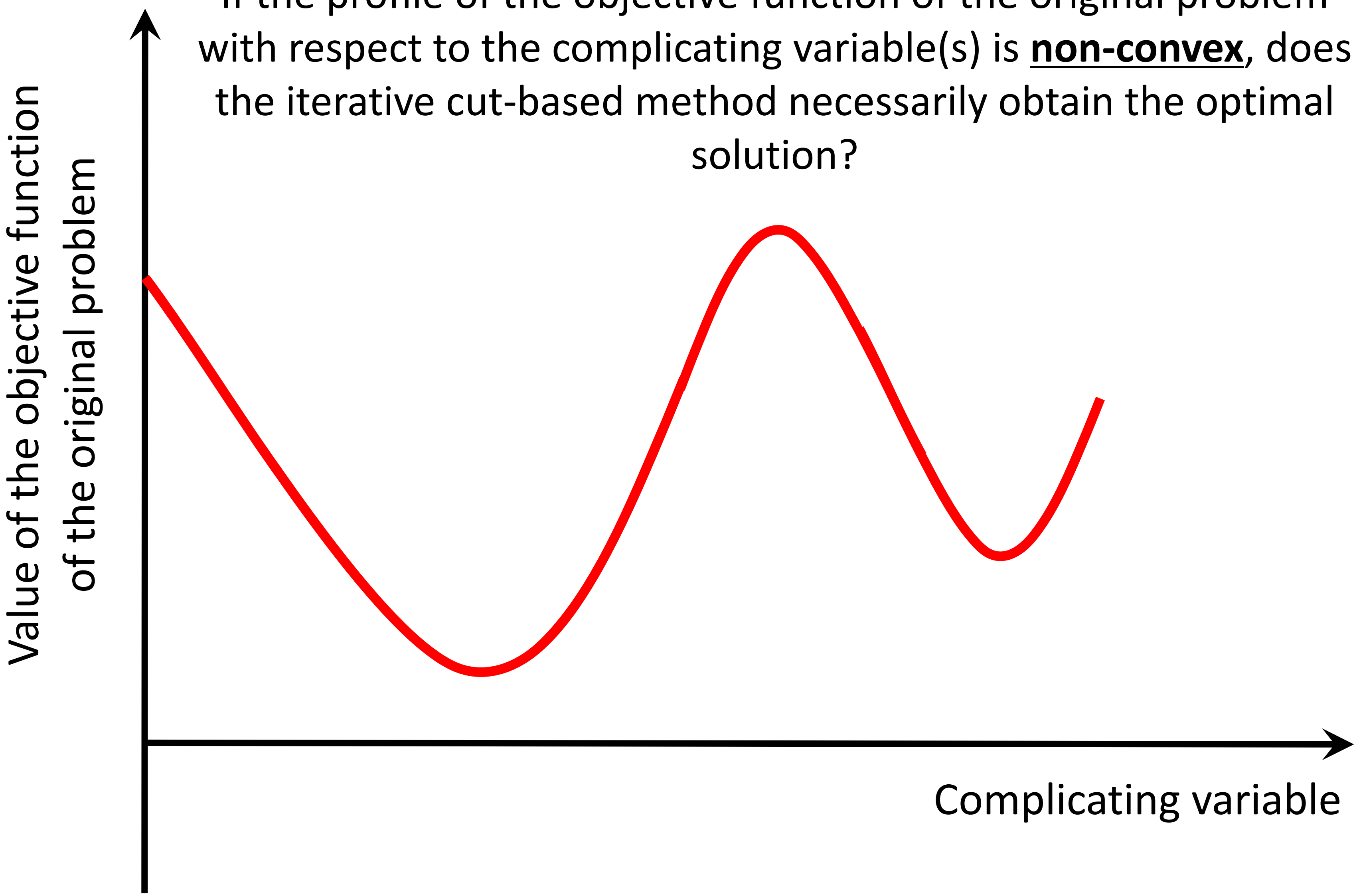
Concept

- We are indeed constructing the original cost profile (dotted line) using cuts!
- More iterations → more cuts → more accurate cost profile!
- This is how Benders' decomposition works!



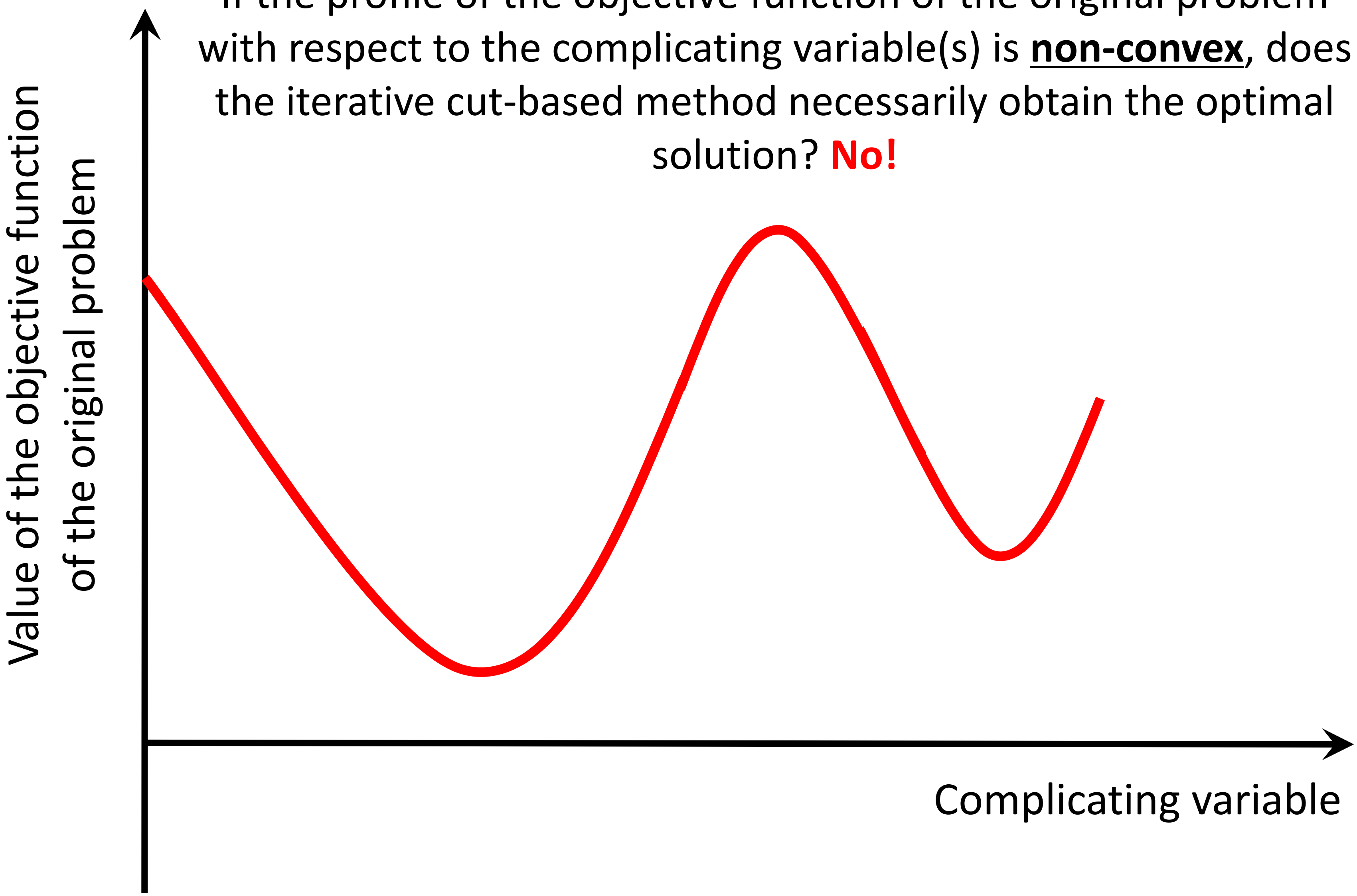
Discussion

If the profile of the objective function of the original problem with respect to the complicating variable(s) is **non-convex**, does the iterative cut-based method necessarily obtain the optimal solution?



Discussion

If the profile of the objective function of the original problem with respect to the complicating variable(s) is **non-convex**, does the iterative cut-based method necessarily obtain the optimal solution? **No!**



References

- M. V. F. Pereira and L. M. V. G. Pinto, "Multi stage stochastic optimization applied to energy planning," *Mathematical Programming,* vol. 52, no. 1-3, pp. 359–375, May 1991.

- A. J. Conejo, E. Castillo, R. Minguez, and R. Garcia-Bertrand, *Decomposition Techniques in Mathematical Programming: Engineering and Science Applications*. Berlin, Germany: Springer, 2006.

Illustrative Example

$$\begin{aligned} \underset{x,y}{\text{minimize}} \quad & z = -y - x/4 \end{aligned}$$

$$\begin{aligned}
y \quad -x &\leq 5 \\
y \quad -\tfrac{1}{2}x &\leq \tfrac{15}{2} \\
y \quad +\tfrac{1}{2}x &\leq \tfrac{35}{2} \\
-y \quad +x &\leq 10 \\
0 &\leq x \leq 16 \\
y &\geq 0\,.
\end{aligned}$$

Illustrative Example

$$\underset{x,y}{\text{minimize}} \quad z = -y - x/4$$

$$\begin{aligned} y \quad -x &\leq 5 \\ y \quad -\tfrac{1}{2}x &\leq \tfrac{15}{2} \\ y \quad +\tfrac{1}{2}x &\leq \tfrac{35}{2} \\ -y \quad +x &\leq 10 \\ 0 \leq x &\leq 16 \\ y &\geq 0\,. \end{aligned}$$

Let's consider *x* as the complicating variable!

Illustrative Example

i : current Benders' iteration
k: set of previous iterations

- For given value for complicating variable (i.e., *x*), subproblem obtains the optimal value for *y* and dual variables (sensitivities).

Subproblem:

$$\underset{y^{(i)}}{\text{minimize}} \ -y^{(i)}$$

subject to

$$y^{(i)} \leq 5 + x^{\text{fixed }(i)} \quad : \pi^{(i)}$$

$$y^{(i)} \leq \frac{15}{2} + \frac{x^{\text{fixed }(i)}}{2} \quad : \mu^{(i)}$$

$$y^{(i)} \leq \frac{35}{2} - \frac{x^{\text{fixed }(i)}}{2} \quad : \sigma^{(i)}$$

$$-y^{(i)} \leq 10 - x^{\text{fixed }(i)} \quad : \gamma^{(i)}$$

$$y^{(i)} \geq 0$$

Illustrative Example

$i:$ current Benders' iteration
$k:$ set of previous iterations

- For given value for complicating variable (i.e., *x*), subproblem obtains the optimal value for *y* and dual variables (sensitivities).

- Based on values for sensitivities achieved in subproblem, master problem obtains the updated value for complicating variable *x*.

Subproblem:

$$\underset{y^{(i)}}{\text{minimize}} \; -y^{(i)}$$

subject to

$$y^{(i)} \leq 5 + x^{\text{fixed}\,(i)} \quad : \pi^{(i)}$$

$$y^{(i)} \leq \frac{15}{2} + \frac{x^{\text{fixed}\,(i)}}{2} \quad : \mu^{(i)}$$

$$y^{(i)} \leq \frac{35}{2} - \frac{x^{\text{fixed}\,(i)}}{2} \quad : \sigma^{(i)}$$

$$-y^{(i)} \leq 10 - x^{\text{fixed}\,(i)} \quad : \gamma^{(i)}$$

$$y^{(i)} \geq 0$$

Master problem:

$$\underset{x^{(i)},\,\alpha^{(i)}}{\text{minimize}} \; -\frac{x^{(i)}}{4} + \alpha^{(i)}$$

subject to

$$0 \leq x^{(i)} \leq 16$$

$$\alpha^{(i)} \geq \alpha^{\text{down}}$$

$$\alpha^{(i)} \geq \pi^{(k)}\left[5 + x^{(i)}\right] + \mu^{(k)}\left[\frac{15}{2} + \frac{x^{(i)}}{2}\right] + \sigma^{(k)}\left[\frac{35}{2} - \frac{x^{(i)}}{2}\right]$$
$$+ \gamma^{(k)}\left[10 - x^{(i)}\right] \quad \forall k = 1, .., i-1$$

Illustrative Example

i : current Benders' iteration
k: set of previous iterations

- For given value for complicating variable (i.e., *x*), subproblem obtains the optimal value for *y* and dual variables (sensitivities).

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$$\underset{y^{(i)}}{\text{minimize}} \; -y^{(i)}$$

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$$y^{(i)} \leq 5 + x^{\text{fixed}\,(i)} \quad : \pi^{(i)}$$

$$y^{(i)} \leq \frac{15}{2} + \frac{x^{\text{fixed}\,(i)}}{2} \quad : \mu^{(i)}$$

$$y^{(i)} \leq \frac{35}{2} - \frac{x^{\text{fixed}\,(i)}}{2} \quad : \sigma^{(i)}$$

$$-y^{(i)} \leq 10 - x^{\text{fixed}\,(i)} \quad : \gamma^{(i)}$$

$$y^{(i)} \geq 0$$

Master problem:

$$\underset{x^{(i)},\,\alpha^{(i)}}{\text{minimize}} \; -\frac{x^{(i)}}{4} + \alpha^{(i)}$$

This is an auxiliary variable, representing the objective function of the subproblem(s) within the objective function of the original problem!

subject to

$$0 \leq x^{(i)} \leq 16$$

$$\alpha^{(i)} \geq \alpha^{\text{down}}$$

$$\alpha^{(i)} \geq \pi^{(k)}\left[5 + x^{(i)}\right] + \mu^{(k)}\left[\frac{15}{2} + \frac{x^{(i)}}{2}\right] + \sigma^{(k)}\left[\frac{35}{2} - \frac{x^{(i)}}{2}\right] + \gamma^{(k)}\left[10 - x^{(i)}\right] \quad \forall k = 1,..,i-1$$

Illustrative Example

i : current Benders' iteration
k: set of previous iterations

- For given value for complicating variable (i.e., *x*), subproblem obtains the optimal value for *y* and dual variables (sensitivities).

- Based on values for sensitivities achieved in subproblem, master problem obtains the updated value for complicating variable *x.*

Subproblem:

$$\underset{y^{(i)}}{\text{minimize}} \; -y^{(i)}$$

subject to

$$y^{(i)} \leq 5 + x^{\text{fixed}\,(i)} \quad : \pi^{(i)}$$

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$$y^{(i)} \leq \frac{35}{2} - \frac{x^{\text{fixed}\,(i)}}{2} \quad : \sigma^{(i)}$$

$$-y^{(i)} \leq 10 - x^{\text{fixed}\,(i)} \quad : \gamma^{(i)}$$

$$y^{(i)} \geq 0$$

Master problem:

$$\underset{x^{(i)},\,\alpha^{(i)}}{\text{minimize}} \; -\frac{x^{(i)}}{4} + \alpha^{(i)}$$

subject to

$$0 \leq x^{(i)} \leq 16$$

$$\alpha^{(i)} \geq \alpha^{\text{down}}$$

$$\alpha^{(i)} \geq \pi^{(k)}\left[5 + x^{(i)}\right] + \mu^{(k)}\left[\frac{15}{2} + \frac{x^{(i)}}{2}\right] + \sigma^{(k)}\left[\frac{35}{2} - \frac{x^{(i)}}{2}\right]$$

$$+\gamma^{(k)}\left[10 - x^{(i)}\right] \quad \forall k = 1,..,i-1$$

To avoid unbounded solution in the first iteration when there is no cut (last constraint). It should be a very large negative constant, e.g., -10^6.

Illustrative Example

i : current Benders' iteration
k: set of previous iterations

- For given value for complicating variable (i.e., *x*), subproblem obtains the optimal value for *y* and dual variables (sensitivities).

- Based on values for sensitivities achieved in subproblem, master problem obtains the updated value for complicating variable *x*.

Subproblem:

$$\underset{y^{(i)}}{\text{minimize}} \; -y^{(i)}$$

subject to

$$y^{(i)} \leq 5 + x^{\text{fixed}\,(i)} \quad : \pi^{(i)}$$

$$y^{(i)} \leq \frac{15}{2} + \frac{x^{\text{fixed}\,(i)}}{2} \quad : \mu^{(i)}$$

$$y^{(i)} \leq \frac{35}{2} - \frac{x^{\text{fixed}\,(i)}}{2} \quad : \sigma^{(i)}$$

$$-y^{(i)} \leq 10 - x^{\text{fixed}\,(i)} \quad : \gamma^{(i)}$$

$$y^{(i)} \geq 0$$

Master problem:

$$\underset{x^{(i)},\,\alpha^{(i)}}{\text{minimize}} \; -\frac{x^{(i)}}{4} + \alpha^{(i)}$$

subject to

$$0 \leq x^{(i)} \leq 16$$

$$\alpha^{(i)} \geq \alpha^{\text{down}}$$

$$\alpha^{(i)} \geq \pi^{(k)}\left[5 + x^{(i)}\right] + \mu^{(k)}\left[\frac{15}{2} + \frac{x^{(i)}}{2}\right] + \sigma^{(k)}\left[\frac{35}{2} - \frac{x^{(i)}}{2}\right]$$
$$+ \gamma^{(k)}\left[10 - x^{(i)}\right] \quad \forall k = 1, \ldots, i-1$$

This constraint generates "Benders' cuts". One new cut in each iteration!

Algorithm

- **Step 0: Initialization**
  Set $i = 1, x^{\text{fixed}\,(1)} = x^{\text{initial}}$, and lower bound (LB) = $-\infty$

- **Step 1: Solve subproblem(s):** obtain the values of all dual variables (sensitivities), and the value of objective function, which is upper bound (UB)

- **Step 2: Convergence check**
  If $|\text{UB} - \text{LB}| \leq \epsilon$, then the optimal solution with a level of accuracy ϵ is obtained, otherwise $i \leftarrow i + 1$

- **Step 3: Solve master problem:** obtain the updated $x^{(i)}$ and the value of $\alpha^{(i)}$ as the updated LB, and go Step 1 with the updated fixed x

GAMS Code (Part 1 / 2)

```
sets
IP iterations /1*100/
alias (IP,K);
parameters
SetX(K)          Set of values obtained for variable x over iterations
SetY(K)          Set of values obtained for variable y over iterations
SetZ(K)          Set of values obtained for subproblem's objective function over iterations
SetDual1(K)      Set of values obtained for dual variable \pi over iterations
SetDual2(K)      Set of values obtained for dual variable \mu over iterations
SetDual3(K)      Set of values obtained for dual variable \sigma over iterations
SetDual4(K)      Set of values obtained for dual variable \gamma over iterations
SetA(K)          Set of values obtained for auxiliary variable \alpha over iterations
;
scalars
d   convergence gap
m   order of current iteration
;
* Initial convergence gap
d=1000;

************ Master problem **************
variable z_down,a,x;
equation  OF,const_master1,const_master2,const_master3,cut;

OF..                z_down =e= -(x/4)+a;
const_master1..          x =l= 16;
const_master2..          x =g= 0;
const_master3..          a =g= -1000;
cut(K)$(m>1 and ord(K)<m)..a =g= [SetDual1(K)*(5+x)]+[SetDual2(K)*(7.5+[x*0.5])]
                                 +[SetDual3(K)*(17.5-[x*0.5])]+[SetDual4(K)*(10-x)];
model master /OF,const_master1,const_master2,const_master3,cut/ ;
```

GAMS Code (Part 2 / 2)

```
************ Subproblem **************
variable z_up,y;
equation Sub_OF,cons_sub1,cons_sub2,cons_sub3,cons_sub4,cons_sub5;
Sub_OF..      z_up  =e= -y;
cons_sub1..       y =l= 5+x.l;
cons_sub2..       y =l= 7.5+(x.l/2);
cons_sub3..       y =l= 17.5-(x.l/2);
cons_sub4..      -y =l= 10-x.l;
cons_sub5..       y =g= 0;
model sub /Sub_OF,cons_sub1,cons_sub2,cons_sub3,cons_sub4,cons_sub5/ ;

************* Iteration **********
loop(IP$(d gt 0.1),
       m=ord(IP);
       SetX(IP)=0; SetY(IP)=0; SetZ(IP)=0; SetDual1(IP)=0; SetDual2(IP)=0;
       SetDual3(IP)=0; SetDual4(IP)=0;
       solve master using lp minimizing z_down;
       solve sub using lp minimizing z_up;
       option lp=cplex;
       d=z_up.l-a.l;
       SetX(IP)=x.l;
       SetY(IP)=y.l;
       SetZ(IP)=z_up.l;
       SetDual1(IP)=cons_sub1.m;
       SetDual2(IP)=cons_sub2.m;
       SetDual3(IP)=cons_sub3.m;
       SetDual4(IP)=cons_sub4.m;
       SetA(IP)=a.l;
);
display x.l,y.l,SetX,SetY,SetA,SetZ,SetDual1,SetDual2,SetDual3,SetDual4;
```

A compact form

Subproblem:

$$\underset{x^{(i)},\, y^{(i)}}{\text{minimize}} \; -y^{(i)}$$

subject to

$$y^{(i)} - x^{(i)} \leq 5 \quad : \pi^{(i)}$$

$$y^{(i)} - \frac{x^{(i)}}{2} \leq \frac{15}{2} \quad : \mu^{(i)}$$

$$y^{(i)} + \frac{x^{(i)}}{2} \leq \frac{35}{2} \quad : \sigma^{(i)}$$

$$-y^{(i)} + x^{(i)} \leq 10 \quad : \gamma^{(i)}$$

$$y^{(i)} \geq 0$$

$$x^{(i)} = x^{\text{fixed}\,(i)} \quad : \rho^{(i)}$$

A compact form

Subproblem:

$$\underset{x^{(i)},\, y^{(i)}}{\text{minimize}} \; -y^{(i)}$$

subject to

$$y^{(i)} - x^{(i)} \leq 5 \quad : \pi^{(i)}$$

$$y^{(i)} - \frac{x^{(i)}}{2} \leq \frac{15}{2} \quad : \mu^{(i)}$$

$$y^{(i)} + \frac{x^{(i)}}{2} \leq \frac{35}{2} \quad : \sigma^{(i)}$$

$$-y^{(i)} + x^{(i)} \leq 10 \quad : \gamma^{(i)}$$

$$y^{(i)} \geq 0$$

$$x^{(i)} = x^{\text{fixed}\,(i)} \quad : \rho^{(i)}$$

In this compact form, there is a constraint for fixing each complicating variable to a given value! However, *x* is now a "variable" in subproblem, though its value is fixed! Only one sensitivity ($\rho^{(i)}$) is needed in the master problem!

A compact form

Subproblem:

$$\underset{x^{(i)},\, y^{(i)}}{\text{minimize}} \; -y^{(i)}$$

subject to

$$y^{(i)} - x^{(i)} \leq 5 \quad : \pi^{(i)}$$

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$$-y^{(i)} + x^{(i)} \leq 10 \quad : \gamma^{(i)}$$

$$y^{(i)} \geq 0$$

$$x^{(i)} = x^{\text{fixed }(i)} \quad : \rho^{(i)}$$

Master problem:

$$\underset{x^{(i)},\, \alpha^{(i)}}{\text{minimize}} \; -\frac{x^{(i)}}{4} + \alpha^{(i)}$$

subject to

$$0 \leq x^{(i)} \leq 16$$

$$\alpha^{(i)} \geq \alpha^{\text{down}}$$

$$\alpha^{(i)} \geq -y^{(k)} + \rho^{(k)}\left[x^{(i)} - x^{(k)}\right] \quad \forall k = 1, .., i-1$$

A compact form

Subproblem:

$$\underset{x^{(i)},\, y^{(i)}}{\text{minimize}} \; -y^{(i)}$$

subject to

$$y^{(i)} - x^{(i)} \leq 5 \quad : \pi^{(i)}$$

$$y^{(i)} - \frac{x^{(i)}}{2} \leq \frac{15}{2} \quad : \mu^{(i)}$$

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A compact formulation for Benders' cut!

GAMS Code for Compact Model (Part 1 / 2)

```
sets
IP iterations /1*100/
alias (IP,K);

parameters
SetX(K)      Set of values obtained for variable x over iterations
SetY(K)      Set of values obtained for variable y over iterations
SetZ(K)      Set of values obtained for subproblem's objective function over iterations
SetDual(K)   Set of values obtained for dual variable \rho over iterations
SetA(K)      Set of values obtained for auxiliary variable \alpha over iterations
;

scalars
d    convergence gap
m    order of current iteration
;

* Initial convergence gap
d=1000;

************ Master problem **************
variable z_down,a,x;
equation  OF,const_master1,const_master2,const_master3,cut;

OF..               z_down =e= -(x/4)+a;
const_master1..        x =l= 16;
const_master2..        x =g= 0;
const_master3..        a =g= -1000;
cut(K)$(m>1 and ord(K)<m)..a =g= -SetY(K)+[SetDual(K)*(x-SetX(K))];

model master /OF,const_master1,const_master2,const_master3,cut/ ;
```

GAMS Code for Compact Model (Part 2 / 2)

```
************ Subproblem **************
variable z_up,x_sub,y;
equation Sub_OF,cons_sub1,cons_sub2,cons_sub3,cons_sub4,cons_sub5,cons_sub6;

Sub_OF..       z_up  =e= -y;
cons_sub1..        y =l= 5+x_sub;
cons_sub2..        y =l= 7.5+(x_sub/2);
cons_sub3..        y =l= 17.5-(x_sub/2);
cons_sub4..       -y =l= 10-x_sub;
cons_sub5..        y =g= 0;
cons_sub6..    x_sub =g= x.l;

model sub /Sub_OF,cons_sub1,cons_sub2,cons_sub3,cons_sub4,cons_sub5,cons_sub6/ ;


************* Iteration **********
loop(IP$(d gt 0.1),
       m=ord(IP);

       SetX(IP)=0;
       SetY(IP)=0;
       SetZ(IP)=0;
       SetDual(IP)=0;

       solve master using lp minimizing z_down;
       solve sub using lp minimizing z_up;
       option lp=cplex;

       d=z_up.l-a.l;

       SetX(IP)=x.l;
       SetY(IP)=y.l;
       SetZ(IP)=z_up.l;
       SetDual(IP)=cons_sub6.m;
       SetA(IP)=a.l;
);

display x.l,y.l,SetX,SetY,SetA,SetZ,SetDual;
```

Exercise 2

Write the formulation of Master and Subproblems (in a compact way) for the following two-stage stochastic optimal power flow (OPF) problem. The two stages are day-ahead (DA) and real-time (RT).

$$\underset{p_g^{\mathrm{DA}},\ p_{g,\omega}^{\mathrm{RT}},\ p_{d,\omega}^{\mathrm{shed}}}{\text{Minimize}} \quad \mathbf{Cost}^{\mathrm{DA}}(p_g^{\mathrm{DA}}) + \mathbb{E}_{\omega}\left[\mathbf{Cost}^{\mathrm{RT}}(p_{g,\omega}^{\mathrm{RT}},\ p_{d,\omega}^{\mathrm{shed}})\right]$$

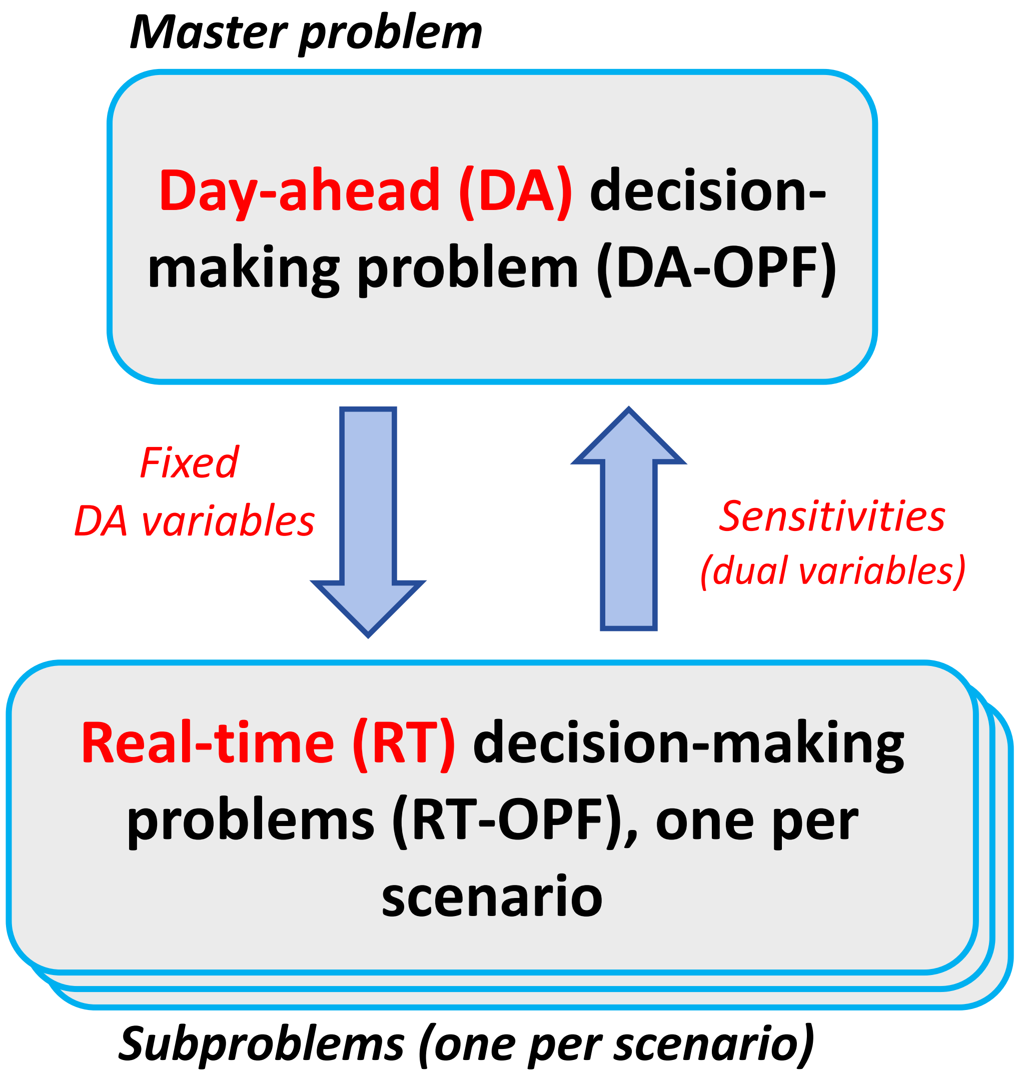
subject to:

$$\mathbf{f}(p_g^{\mathrm{DA}}) \leq 0$$

$$\mathbf{g}(p_{g,\omega}^{\mathrm{RT}},\ p_{d,\omega}^{\mathrm{shed}}) \leq 0 \qquad \forall \omega$$

$$\mathbf{h}(p_g^{\mathrm{DA}},\ p_{g,\omega}^{\mathrm{RT}},\ p_{d,\omega}^{\mathrm{shed}}) \leq 0 \qquad \forall \omega$$

Exercise 2



Master problem
Day-ahead (DA) decision-making problem (DA-OPF)
Fixed DA variables
Sensitivities (dual variables)
Real-time (RT) decision-making problems (RT-OPF), one per scenario
Subproblems (one per scenario)

Are you keen on mathematical background behind Benders' decomposition?

Are you keen on mathematical background behind Benders' decomposition?

This is not to be covered now

(please check it at home)!

Mathematical background

Consider the following two-stage deterministic problem:

$$\min_{x_1, x_2} \quad c_1 x_1 + c_2\, x_2$$

subject to

$$A_1 x_1 \geq b_1$$

$$E_1 x_1 + A_2 x_2 \geq b_2$$

Mathematical background

Consider the following two-stage deterministic problem:

First-stage cost

$$\min_{x_1, x_2} \underbrace{c_1 x_1}_{\text{First-stage cost}} + \underbrace{c_2 \, x_2}_{\text{Second-stage cost}}$$

Second-stage cost

subject to

$$A_1 x_1 \geq b_1$$ ← First-stage constraint

$$E_1 x_1 + A_2 x_2 \geq b_2$$

Linking constraint

Mathematical background

Consider the following two-stage deterministic problem:

First-stage cost

$$\min_{x_1, x_2} \; \underbrace{c_1 x_1}_{\text{First-stage cost}} + \underbrace{c_2 \, x_2}_{\text{Second-stage cost}}$$

Second-stage cost

subject to

$$A_1 x_1 \geq b_1$$ ← First-stage constraint

$$E_1 x_1 + A_2 x_2 \geq b_2$$

Linking constraint

Let's try to solve the first- and second-stage problems separately!

Mathematical background

Consider the following two-stage deterministic problem:

$$\min_{x_1,x_2} \; c_1 x_1 + c_2\, x_2$$

subject to

$$A_1 x_1 \geq b_1$$

$$E_1 x_1 + A_2 x_2 \geq b_2$$

First-stage problem:

$$\min_{x_1} \; c_1 x_1 + \alpha_1(x_1)$$

subject to

$$A_1 x_1 \geq b_1$$

Second-stage problem:

$$\alpha_1(x_1) = \min_{x_2} \; c_2 x_2$$

subject to

$$A_2 x_2 \geq b_2 - E_1 x_1$$

$\alpha_1(x_1)$: the second-stage cost as a function of the first-stage decisions x_1 (future cost function)

Mathematical background

Consider the following two-stage deterministic problem:

First-stage problem:

$$\min_{x_1} \; c_1 x_1 + \alpha_1(x_1)$$

subject to

$$A_1 x_1 \geq b_1$$

Second-stage problem:

$$\alpha_1(x_1) = \min_{x_2} \; c_2 x_2$$

subject to

$$A_2 x_2 \geq b_2 - E_1 x_1$$

<u>Note</u>: x_1 appears in the second-stage problem! So, these two problems cannot be still solved separately!

Mathematical background

Consider the following two-stage deterministic problem:

First-stage problem:

$$\min_{x_1} \quad c_1 x_1 + \alpha_1(x_1)$$

subject to

$$A_1 x_1 \geq b_1$$

Second-stage problem:

$$\alpha_1(x_1) = \min_{x_2} \quad c_2 x_2$$

subject to

$$A_2 x_2 \geq b_2 - E_1 x_1$$

Any potential approach to solve the first- and second-stage problems separately?

Mathematical background

- **Step 1)** **Discretize** x_1 into a set of trial values $\{\hat{x}_{1i}, i = 1, \dots, n\}$
- **Step 2)** Solve the second-stage problem for each trial value
- **Step 3)** Construct future cost function $\alpha_1(x_1)$. Intermediate values of $\alpha_1(x_1)$ are obtained by interpolation from the neighboring discretized values.
- **Step 4)** Solve the first-stage problem using the future cost function constructed.

First-stage problem:

$$\min_{x_1} \; c_1 x_1 + \alpha_1(x_1)$$

$$\text{subject to}$$

$$A_1 x_1 \geq b_1$$

Second-stage problem:

$$\alpha_1(x_1) = \min_{x_2} \; c_2 x_2$$

$$\text{subject to}$$

$$A_2 x_2 \geq b_2 - E_1 x_1$$

Mathematical background

- **Step 1)** Discretize x_1 into a set of trial values $\{\hat{x}_{1i}, i = 1, \ldots, n\}$
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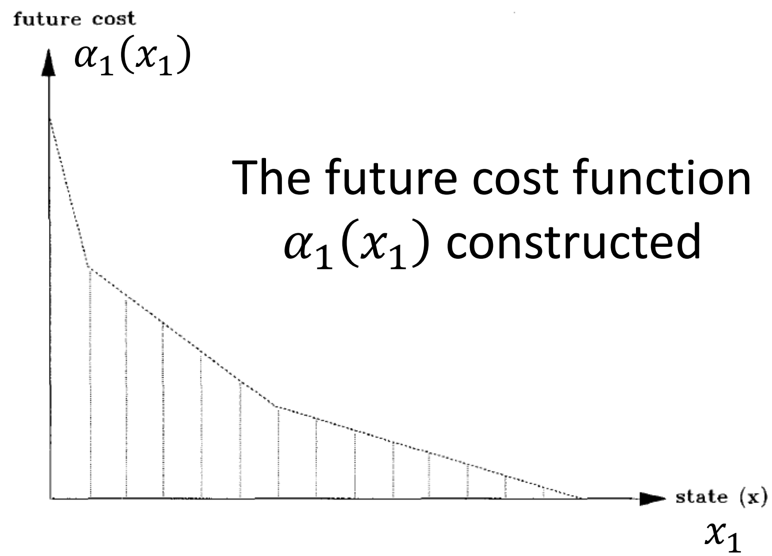

The future cost function $\alpha_1(x_1)$ constructed

Mathematical background

- **Step 1)** Discretize x_1 into a set of trial values $\{\hat{x}_{1i}, i = 1, \ldots, n\}$
- **Step 2)** Solve the second-stage problem for each trial value
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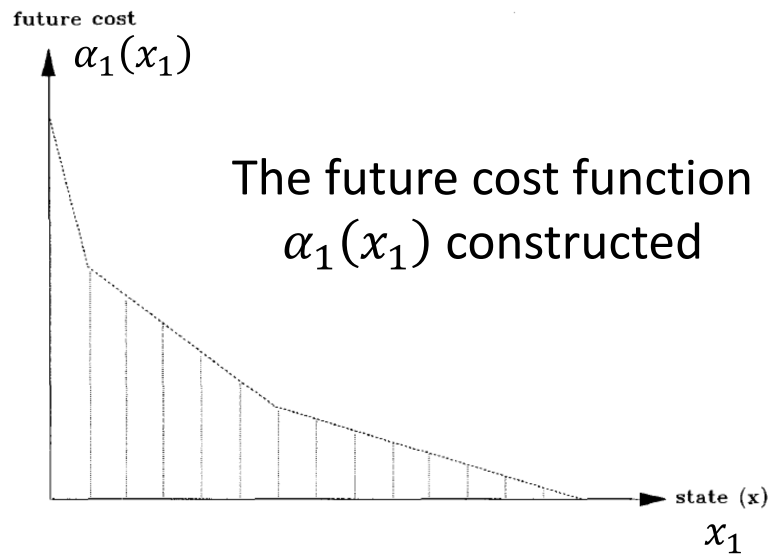

What is this technique?

Mathematical background

- **Step 1)** **Discretize** x_1 into a set of trial values $\{\hat{x}_{1i}, i = 1, \ldots, n\}$
- **Step 2)** Solve the second-stage problem for each trial value
- **Step 3)** Construct future cost function $\alpha_1(x_1)$. Intermediate values of $\alpha_1(x_1)$ are obtained by interpolation from the neighboring discretized values.
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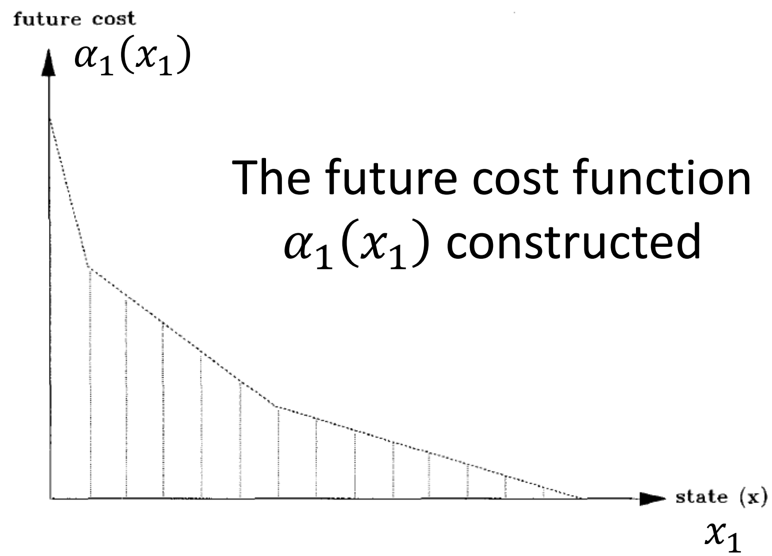
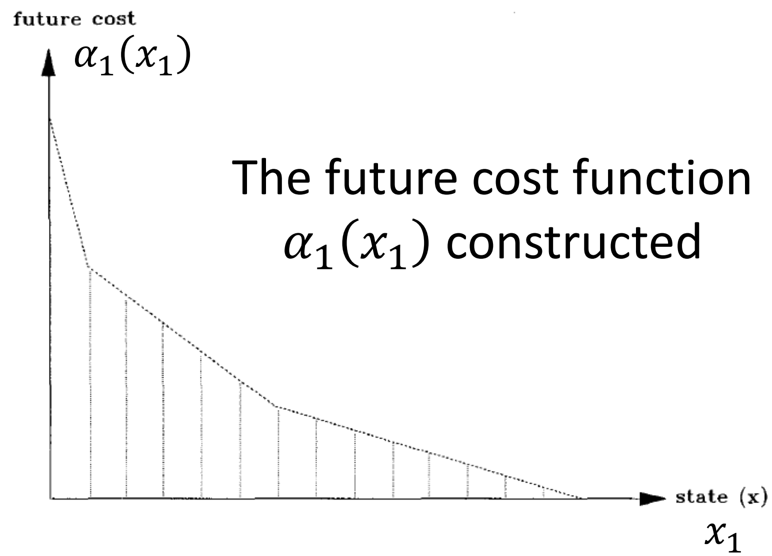


The future cost function $\alpha_1(x_1)$ constructed

What is this technique?

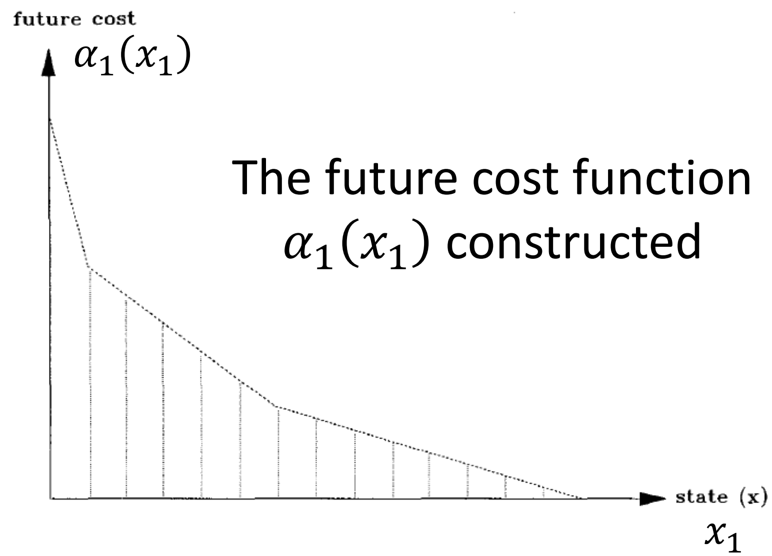
Dynamic programming (DP)

Mathematical background



What is the main drawback of dynamic programming (DP)?

Mathematical background



What is the main drawback of dynamic programming (DP)?

DP needs to discretize the decision variables x_1, which results in computational issues!

For example, 10 decision variables and 4 discretized value for each variable leads to 4^{10} discrete values!

Mathematical background

Any alternative solution approach?

Mathematical background

Dual dynamic programming (DDP) instead of DP!

Advantage:

To approximate the future cost function $\alpha_1(x_1)$ by analytical functions rather than a set of discrete values!

Dual dynamic programming (DDP)

Second-stage problem:

$$\alpha_1(x_1) = \min_{x_2} \; c_2 x_2$$

subject to

$$A_2 x_2 \geq b_2 - E_1 x_1 \quad : \pi$$

Dual variable

Dual dynamic programming (DDP)

Second-stage problem:

$$\alpha_1(x_1) = \min_{x_2} \; c_2 x_2$$

subject to

$$A_2 x_2 \geq b_2 - E_1 x_1 \quad : \pi$$

Dual of the second-stage problem:

$$\max_{\pi} \; \pi(b_2 - E_1 x_1)$$

subject to

$$c_2 \geq \pi A_2$$

Dual dynamic programming (DDP)

Second-stage problem:

$$\alpha_1(x_1) = \min_{x_2} \; c_2 x_2$$

subject to

$$A_2 x_2 \geq b_2 - E_1 x_1 \quad : \pi$$

Dual of the second-stage problem:

$$\max_{\pi} \; \pi(b_2 - E_1 x_1)$$

subject to

$$c_2 \geq \pi A_2$$

$$\alpha_1(x_1) = \pi^*(b_2 - E_1 x_1)$$

In the optimal solution
(strong duality theorem)

Dual dynamic programming (DDP)

Second-stage problem:

$$\alpha_1(x_1) = \min_{x_2} \; c_2 x_2$$

subject to

$$A_2 x_2 \geq b_2 - E_1 x_1 \quad : \pi$$

Dual of the second-stage problem:

$$\max_{\pi} \; \pi(b_2 - E_1 x_1)$$

subject to

$$c_2 \geq \pi A_2$$

$$\alpha_1(x_1) = \pi^*(b_2 - E_1 x_1)$$

In the optimal solution (strong duality theorem)

Interpretation: there is a linear relation between x_1 and the future cost function $\alpha_1(x_1)$ if the sensitivity π^* is known! x_2 has not appeared.

Dual dynamic programming (DDP)

$$\alpha_1(x_1) = \pi^*(b_2 - E_1 x_1)$$

In the optimal solution

(strong duality theorem: value of primal objective function = value of dual objective function)

Dual dynamic programming (DDP)

$$\alpha_1(x_1) = \pi^*(b_2 - E_1 x_1)$$

In the optimal solution

(strong duality theorem: value of primal objective function = value of dual objective function)

$$\alpha_1(x_1) \geq \pi\,(b_2 - E_1 x_1)$$

In any feasible (not necessarily optimal) point within feasible space

(weak duality theorem: value of primal objective function $\geq$ value of dual objective function)

Dual dynamic programming (DDP)

$$\alpha_1(x_1) = \pi^*(b_2 - E_1 x_1)$$

In the optimal solution

(strong duality theorem: value of primal objective function = value of dual objective function)

$$\alpha_1(x_1) \geq \pi\,(b_2 - E_1 x_1)$$

In any feasible (not necessarily optimal) point within feasible space

(weak duality theorem: value of primal objective function ≥ value of dual objective function)

Challenge: we do not have the optimal value for dual variable π^*!

Dual dynamic programming (DDP)

$$\alpha_1(x_1) = \pi^*(b_2 - E_1 x_1)$$

Assume π^1, π^2, …, π^n are possible values for π. Let's assume the optimal value for dual variable, i.e., π^*, is included in this set of possible values. Then, $\alpha_1(x_1)$ can be characterized as follows:

Dual dynamic programming (DDP)

$$\alpha_1(x_1) = \pi^*(b_2 - E_1 x_1)$$

Assume π^1, π^2, …, π^n are possible values for π. Let's assume the optimal value for dual variable, i.e., π^*, is included in this set of possible values. Then, $\alpha_1(x_1)$ can be characterized as follows:

$$\alpha_1(x_1) = \min_{\alpha} \ \alpha$$

subject to

$$\alpha \geq \pi^1(b_2 - E_1 x_1)$$

$$\alpha \geq \pi^2(b_2 - E_1 x_1)$$

.

.

.

$$\alpha \geq \pi^n(b_2 - E_1 x_1)$$

Recall:

- In non-optimal points, the value of α representing the primal objective function ($\alpha_1(x_1)$) is great than the dual objective function (weak duality)!
- The values of those objective functions are equal in the optimal point (strong duality)!

Dual dynamic programming (DDP)

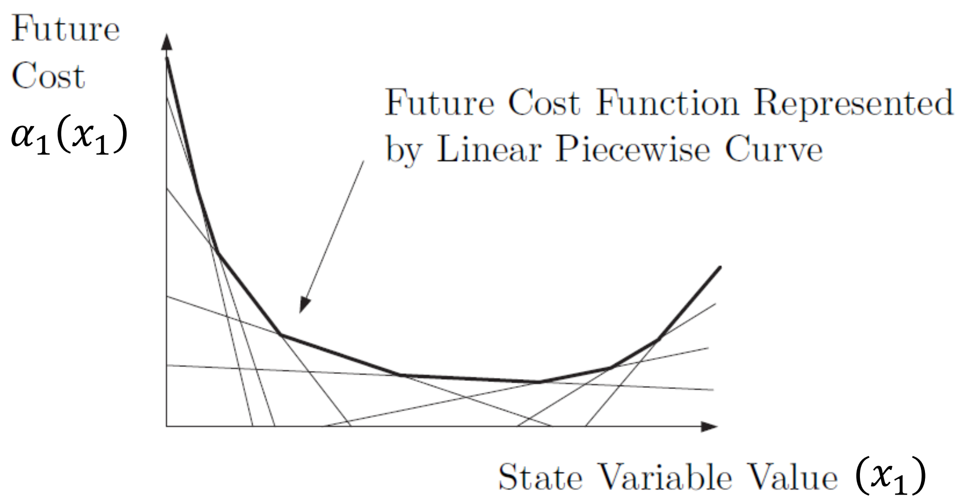
$$\alpha_1(x_1) = \pi^*(b_2 - E_1 x_1)$$

Assume π^1, π^2, …, π^n are possible values for π. Let's assume the optimal value for dual variable, i.e., π^*, is included in this set of possible values. Then, $\alpha_1(x_1)$ can be characterized as follows:

$$\alpha_1(x_1) = \min_{\alpha} \ \alpha$$

subject to

$$\alpha \geq \pi^1(b_2 - E_1 x_1)$$
$$\alpha \geq \pi^2(b_2 - E_1 x_1)$$
$$\vdots$$
$$\alpha \geq \pi^n(b_2 - E_1 x_1)$$



Note: this means that we can construct a piecewise linear function for $\alpha_1(x_1)$ problem (analytically but approximately) without need to discretize x_1!

Dual dynamic programming (DDP)

$$\alpha_1(x_1) = \pi^*(b_2 - E_1 x_1)$$

Assume π^1, π^2, …, π^n are possible values for π. Let's assume the optimal value for dual variable, i.e., π^*, is included in this set of possible values. Then, $\alpha_1(x_1)$ can be characterized as follows:

$$\alpha_1(x_1) = \min_{\alpha} \ \alpha$$

subject to

$$\alpha \geq \pi^1(b_2 - E_1 x_1)$$

$$\alpha \geq \pi^2(b_2 - E_1 x_1)$$

.

.

.

$$\alpha \geq \pi^n(b_2 - E_1 x_1)$$

Recall the first-stage problem:

$$\min_{x_1} \ c_1 x_1 + \alpha_1(x_1)$$

subject to

$$A_1 x_1 \geq b_1$$

Dual dynamic programming (DDP)

$$\alpha_1(x_1) = \pi^*(b_2 - E_1 x_1)$$

Assume π^1, π^2, …, π^n are possible values for π. Let's assume the optimal value for dual variable, i.e., π^*, is included in this set of possible values. Then, $\alpha_1(x_1)$ can be characterized as follows:

$$\alpha_1(x_1) = \min_{\alpha} \ \alpha$$

subject to

$$\alpha \geq \pi^1(b_2 - E_1 x_1)$$

$$\alpha \geq \pi^2(b_2 - E_1 x_1)$$

.

.

.

$$\alpha \geq \pi^n(b_2 - E_1 x_1)$$

Recall the first-stage problem:

$$\min_{x_1} \ c_1 x_1 + \alpha_1(x_1)$$

subject to

$$A_1 x_1 \geq b_1$$

Let's merge them!

Dual dynamic programming (DDP)

The updated first-stage problem including the piecewise linear function $\alpha_1(x_1)$

$$\min_{\alpha,\, x_1} \; c_1 x_1 + \alpha$$

subject to

$$A_1 x_1 \geq b_1$$

$$\alpha \geq \pi^1 (b_2 - E_1 x_1)$$

$$\alpha \geq \pi^2 (b_2 - E_1 x_1)$$

$$\vdots$$

$$\alpha \geq \pi^n (b_2 - E_1 x_1)$$

Dual dynamic programming (DDP)

The updated first-stage problem including the piecewise linear function $\boldsymbol{\alpha_1(x_1)}$

$$\min_{\alpha, x_1} \; c_1 x_1 + \alpha$$

subject to

$$A_1 x_1 \geq b_1$$

$$\alpha \geq \pi^1 (b_2 - E_1 x_1)$$

$$\alpha \geq \pi^2 (b_2 - E_1 x_1)$$

$$\vdots$$

$$\alpha \geq \pi^n (b_2 - E_1 x_1)$$

- This means that we could solve the first-stage problem (including variable x_1 only) independent from the second-stage problem (which includes variable x_2)!
- **Assumption:** we have the set of possible values for dual variable π^*!

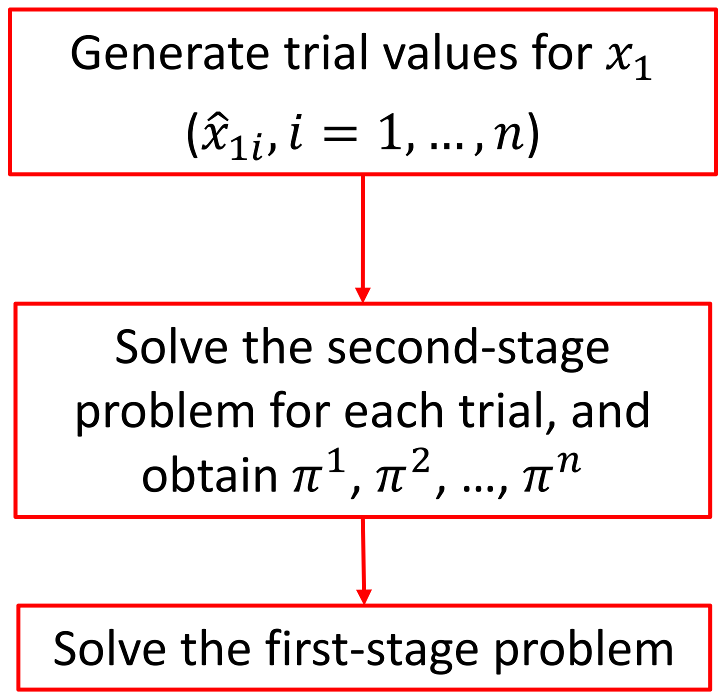
Dual dynamic programming (DDP)

How to generate possible values for π^*, i.e., $\pi^1, \pi^2, \ldots, \pi^n$?

Dual dynamic programming (DDP)

How to generate possible values for π^*, i.e., $\pi^1, \pi^2, \ldots, \pi^n$?

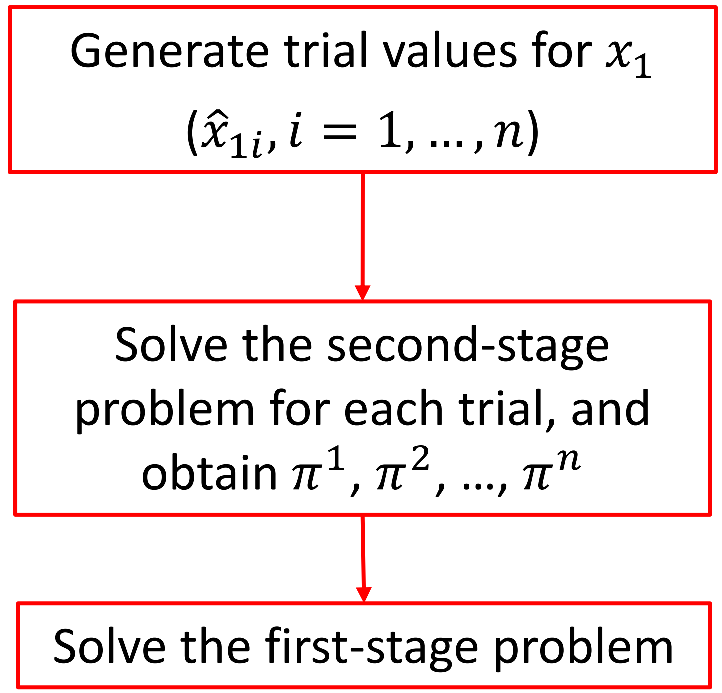
Option 1:



Dual dynamic programming (DDP)

How to generate possible values for π^*, i.e., $\pi^1, \pi^2, \ldots, \pi^n$?

Option 1:



Do you recommend this option?

Dual dynamic programming (DDP)

How to generate possible values for π^*, i.e., π^1, π^2, ..., π^n ?

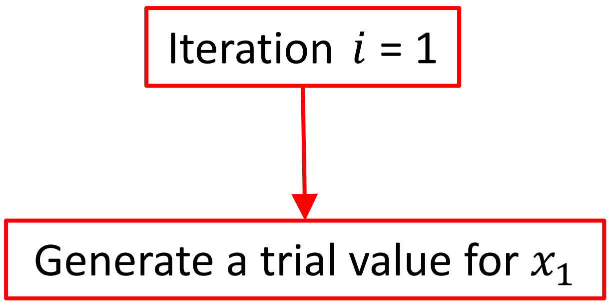
Option 2 (a systematic iterative approach):

Iteration i = 1

Dual dynamic programming (DDP)

How to generate possible values for π^*, i.e., $\pi^1, \pi^2, \ldots, \pi^n$?

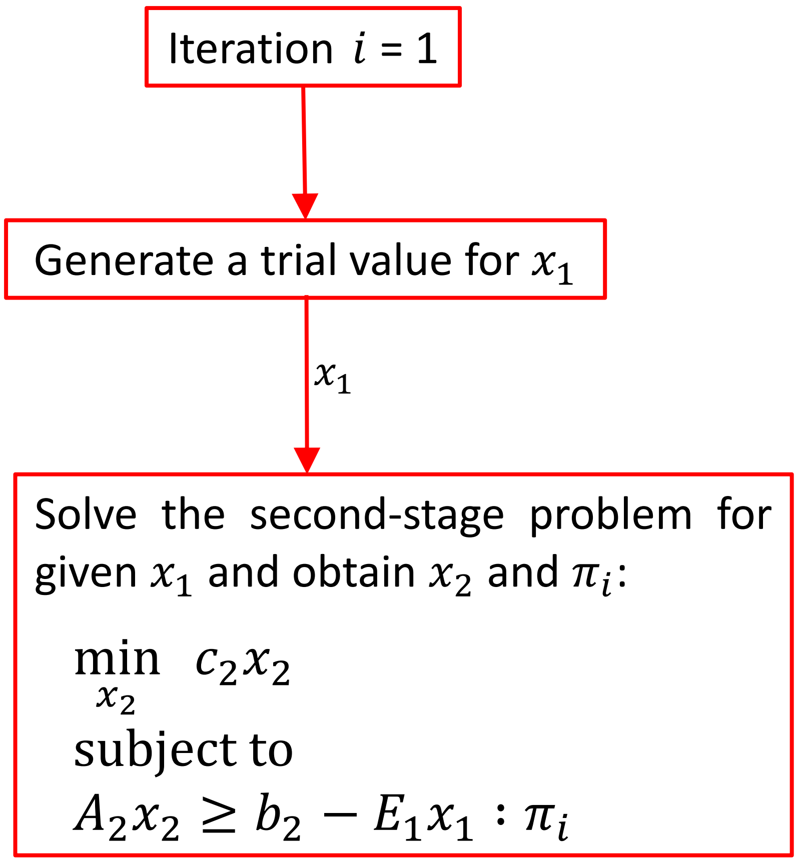
Option 2 (a systematic iterative approach):



Dual dynamic programming (DDP)

How to generate possible values for π^*, i.e., $\pi^1, \pi^2, \ldots, \pi^n$?

Option 2 (a systematic iterative approach):



Dual dynamic programming (DDP)

How to generate possible values for π^*, i.e., $\pi^1, \pi^2, \ldots, \pi^n$?

Option 2 (a systematic iterative approach):

Iteration i = 1

Generate a trial value for x_1

x_1

Solve the second-stage problem for given x_1 and obtain x_2 and π_i:

$$\min_{x_2} \; c_2 x_2$$

subject to

$$A_2 x_2 \geq b_2 - E_1 x_1 \; : \pi_i$$

π_i

Solve the first-stage problem for given π_i and update x_1:

$$\min_{\alpha, x_1} \; c_1 x_1 + \alpha$$

subject to

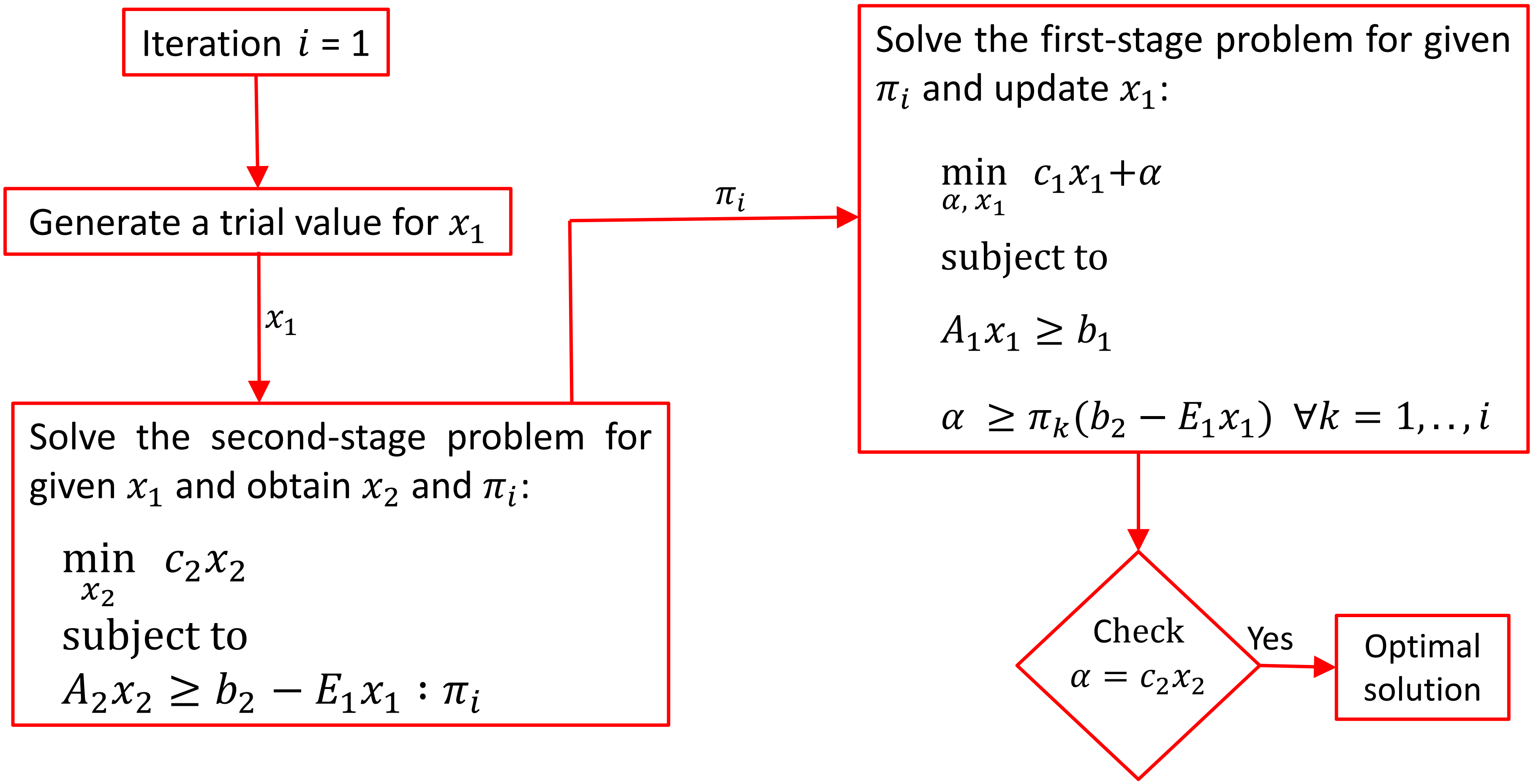
$$A_1 x_1 \geq b_1$$

$$\alpha \; \geq \pi_k (b_2 - E_1 x_1) \;\; \forall k = 1, .., i$$

Dual dynamic programming (DDP)

How to generate possible values for π^*, i.e., $\pi^1, \pi^2, \ldots, \pi^n$?

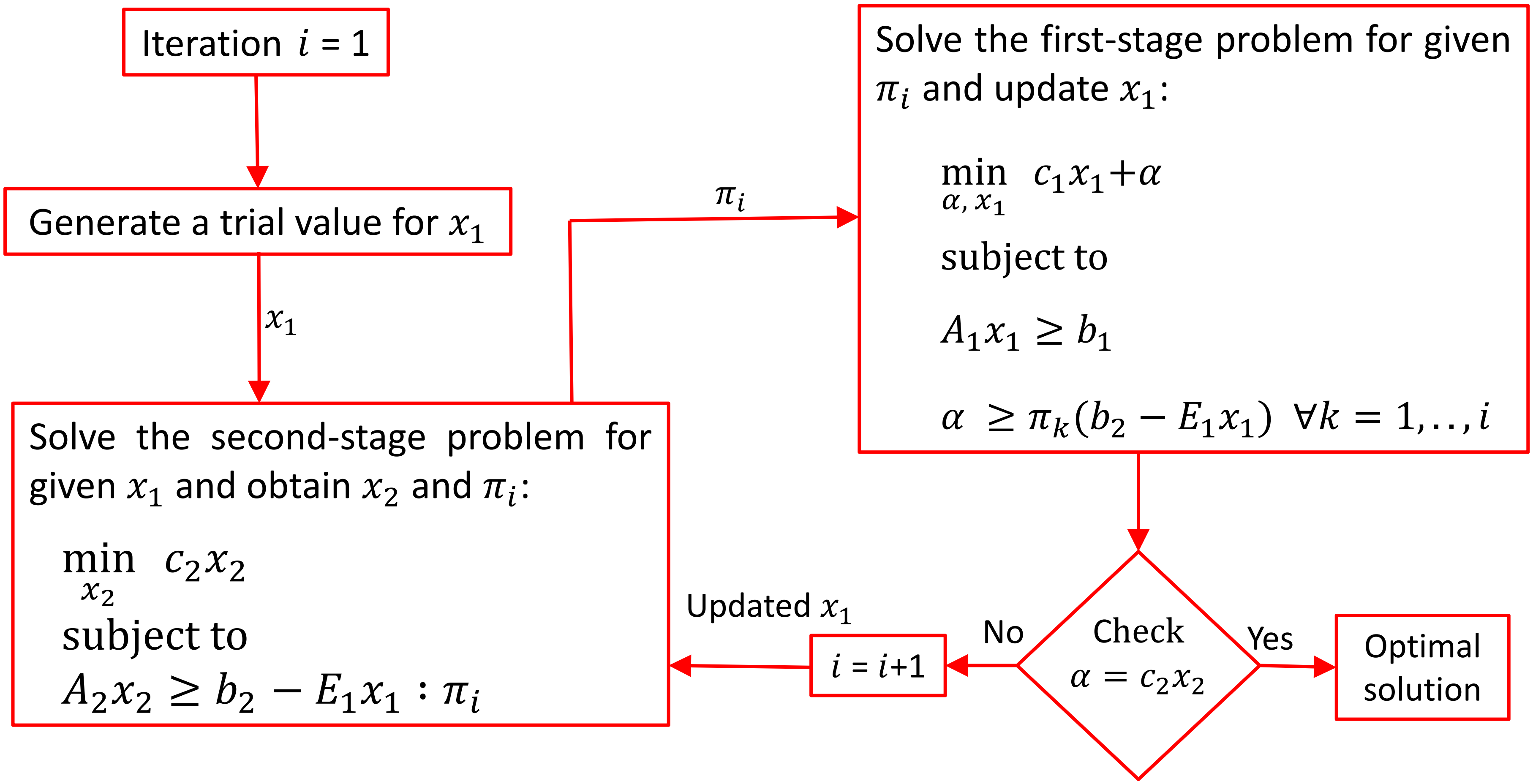
Option 2 (a <u>systematic iterative</u> approach):



Dual dynamic programming (DDP)

How to generate possible values for π^*, i.e., $\pi^1, \pi^2, \ldots, \pi^n$?

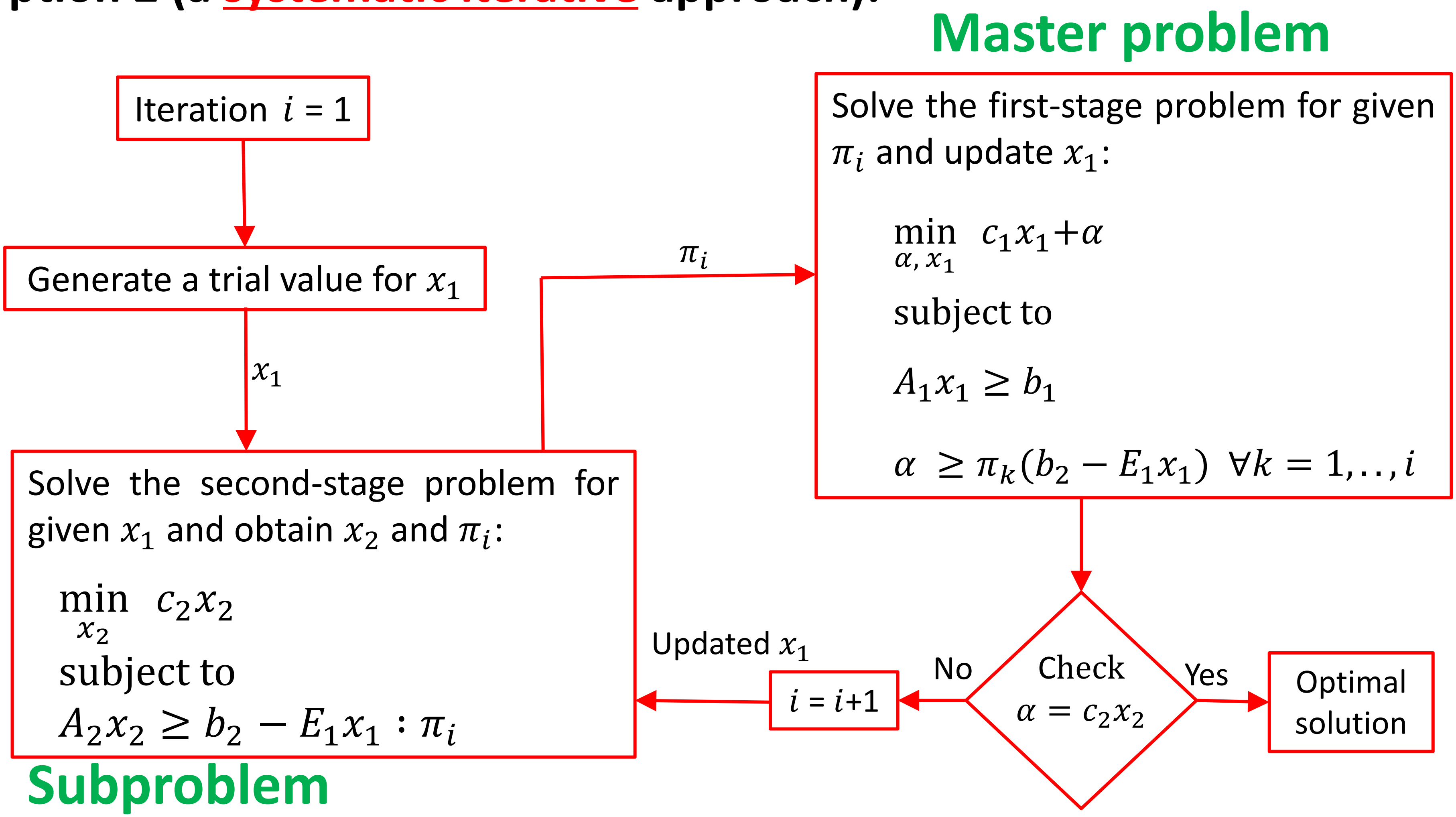
Option 2 (a systematic iterative approach):



Dual dynamic programming (DDP)

How to generate possible values for π^*, i.e., $\pi^1, \pi^2, \ldots, \pi^n$?

Option 2 (a systematic iterative approach):



Dual dynamic programming (DDP)

This solution method (i.e., iterative systematic DDP) is indeed Benders' decomposition approach!

Thanks for your attention!

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