Introduction to convex relaxations of AC-OPF

Semidefinite Programming and OPF

DTU Summer School 2018

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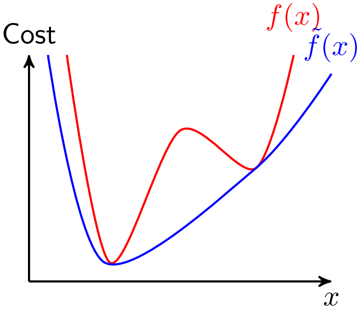
The Goals for this Lecture!

- Why convex relaxations and SDP?
- Semidefinite Programming (SDP)
- Convex Relaxations for AC-OPF

Convexifying the Optimal Power Flow problem (OPF)

- Convex relaxations transform the OPF to a convex Semi-Definite Program (SDP)
- Under certain conditions, the obtained solution is the global optimum to the original OPF problem[1]



Convex Relaxation

[1] Javad Lavaei and Steven H Low. "Zero duality gap in optimal power flow problem". In: *IEEE Transactions on Power Systems* 27.1 (2012), pp. 92–107

Outline of Lecture

- Motivation: Convex vs. Non-Convex Problem and SDP
- **What is SDP?**
  - Numerical Example
- What is a Positive Semidefinite Matrix?
- SDP vs. LP
- Convex Relaxations for AC-OPF

Semidefinite Programming

Semidefinite programming (SDP) is the most exciting development in the mathematical programming in the 1990's [2]

- Between 2008-2012 we had the first formulations (and breakthroughs) for a convexified AC-OPF problem.

[2] Robert M. Freund, Introduction to Mathematical Programming, MIT Lecture Notes, 2009

What is Semidefinite Programming? (SDP)

- SDP is the "generalized" form of an LP (linear program)

Linear Programming

$$\min c^T \cdot x$$

subject to:

$$a_i \cdot x = b_i, \quad i = 1, \ldots, m$$
$$x \geq 0, \quad x \in R^n$$

Semidefinite Programming

$$\min C \bullet X := \sum_i \sum_j C_{ij} X_{ij}$$

subject to:

$$A_i \bullet X = b_i, \quad i = 1, \ldots, m$$
$$X \succeq 0$$

- LP: Optimization variables in the form of a vector x.
- SDP: Optim. variables in the form of a positive semidefinite *matrix* X.

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Positive Semidefinite Matrix??

Ignore it for now. We will come back to it in a few slides.

$C \bullet X := \sum_i \sum_j C_{ij} X_{ij}$ – What's that?

$C \bullet X$: "*sum* of elementwise multiplication"

$$\min \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \bullet \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$$

$$\min C \bullet X := \sum_i \sum_j C_{ij} X_{ij}$$

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$$\begin{bmatrix} A1_{11} & A1_{12} \\ A1_{21} & A1_{22} \end{bmatrix} \bullet \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} = b_1$$

$$\begin{bmatrix} A2_{11} & A2_{12} \\ A2_{21} & A2_{22} \end{bmatrix} \bullet \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} = b_2$$

$$\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \succeq 0$$

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$$\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \succeq 0$$

$$\min c_{11}X_{11} + c_{12}X_{12} + c_{21}X_{21} + c_{22}X_{22}$$

subject to:

$$A1_{11}X_{11} + A1_{12}X_{12} + A1_{21}X_{21} + A1_{22}X_{22} = b_1$$

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$$\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \succeq 0$$

SDP vs LP

Semidefinite Programming

$$\min \; c_{11}X_{11} + c_{12}X_{12} + c_{21}X_{21} + c_{22}X_{22}$$

subject to:

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$$\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \succeq 0$$

Linear Program

LP $\rightarrow$ Optim. variables in vector:

$X = [X_{11}\ X_{12}\ X_{21}\ X_{22}]^T$

$$\min c^T \cdot X$$

subject to:

$$A_1 \cdot X = b_1$$
$$A_2 \cdot X = b_2$$
$$X_{11} \geq 0,\ X_{12} \geq 0,$$
$$X_{21} \geq 0,\ X_{22} \geq 0$$

- SDP looks very much like a LP!
- Only difference: instead of each element of $\mathbf{X}$ to be positive, $\mathbf{X}$ must be a positive semidefinite matrix!

Numerical Example

- Assume X is a 3×3 matrix.

$$A_1 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 3 & 7 \\ 1 & 7 & 5 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 2 & 8 \\ 2 & 6 & 0 \\ 8 & 0 & 4 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 9 & 0 \\ 3 & 0 & 7 \end{bmatrix}$$

$$b_1 = 11 \qquad b_2 = 19$$

Formulate the optimization problems w.r.t. to the elements of matrix X, i.e. linear equations w.r.t. X_{11}, X_{12}, etc.

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Answer in p.6 of R. Freund, Introduction to Semidefinite Programming, MIT Lecture Notes, 2009. `https://ocw.mit.edu/courses/electrical-engineering-and-computer-science/6-251j-introduction-to-mathematical-programming-fall-2009/readings/MIT6_251JF09_SDP.pdf`

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- **What is a Positive Semidefinite Matrix?**
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What is a Positive Semidefinite Matrix P?

- P must be symmetric

P is a positive semidefinite matrix iff:

- $x^T P x \geq 0$, for *any* non-zero vector x

  or

- $eig(P) \geq 0$ for *all* eigenvalues of P

  or

- all leading principal minors are non-negative

What are Principal Minors?

Principal minors are the determinants of submatrices of P. To obtain a principal minor of order k, you delete the same $n-k$ rows and $n-k$ columns. The leading principal minor of order k is the minor of order k obtained by deleting the *last* $n-k$ rows and columns[3]

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \Rightarrow$$

1st order: $\boldsymbol{p_{11}}$, p_{22} 2nd order: $\begin{vmatrix} \boldsymbol{p_{11}} & \boldsymbol{p_{12}} \\ \boldsymbol{p_{21}} & \boldsymbol{p_{22}} \end{vmatrix}$

$$P = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix} \Rightarrow$$

1st order: $\boldsymbol{p_{11}}$, p_{22}, p_{33}

2nd order:

$$\begin{vmatrix} p_{22} & p_{23} \\ p_{32} & p_{33} \end{vmatrix} \quad \begin{vmatrix} p_{11} & p_{13} \\ p_{31} & p_{33} \end{vmatrix} \quad \begin{vmatrix} \boldsymbol{p_{11}} & \boldsymbol{p_{12}} \\ \boldsymbol{p_{21}} & \boldsymbol{p_{22}} \end{vmatrix}$$

3rd order: $\begin{vmatrix} \boldsymbol{p_{11}} & \boldsymbol{p_{12}} & \boldsymbol{p_{13}} \\ \boldsymbol{p_{21}} & \boldsymbol{p_{22}} & \boldsymbol{p_{23}} \\ \boldsymbol{p_{31}} & \boldsymbol{p_{32}} & \boldsymbol{p_{33}} \end{vmatrix}$

[3] In bold are the leading principal minors. You can find helpful info here: `http://www.dr-eriksen.no/teaching/GRA6035/2010/lecture5.pdf`.

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SDP vs LP

Does it make such a difference if we optimize over a positive semidefinite X instead of having all individual elements of this matrix positive?

Yes!

SDP vs LP variables: Example

$$X = \begin{bmatrix} x_2 & x_1 \\ x_1 & 1 \end{bmatrix} \succeq 0$$

When is P positive semidefinite?

SDP vs LP variables: Example

$$X = \begin{bmatrix} x_2 & x_1 \\ x_1 & 1 \end{bmatrix} \succeq 0$$

When is P positive semidefinite?

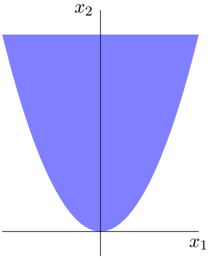
For X to be positive semidefinite, it must be:

- X symmetric $\rightarrow$ OK!
- first order princ.minor positive: $1 > 0 \rightarrow$ OK!
- second order princ.minor positive: $x_2 - x_1^2 \geq 0$

Example: Feasible space of SDP vs LP variables

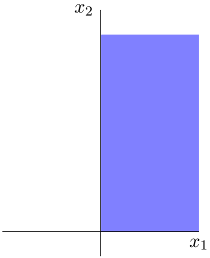
SDP

$$X = \begin{bmatrix} x_2 & x_1 \\ x_1 & 1 \end{bmatrix} \succeq 0 \Rightarrow x_2 - x_1^2 \geq 0$$



LP

$$x_1 \geq 0$$
$$x_2 \geq 0$$



Example: Feasible space of SDP vs LP variables

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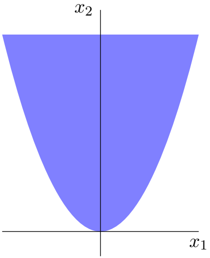

LP

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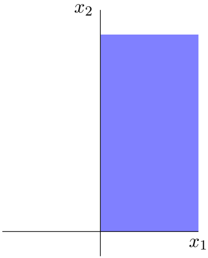

- In SDP we can express quadratic constraints, e.g. x_1^2 or $x_1 x_2$
- In general, in SDP we allow the variables to "move" in a larger space $\rightarrow$ here, x_1 can take negative values
- SDP applies to a larger family of problems $\rightarrow$ LP special case of SDP

LP as a special case of SDP

$$\min \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \bullet \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$$

subject to:

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$$\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \succeq 0$$

How should C, A_1, A_2 look like so that our SDP problem become an LP?

Assume that the LP will only have two variables.

LP as a special case of SDP

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How should C, A_1, A_2 look like so that our SDP problem become an LP?

Assume that the LP will only have two variables.

Answer: If C, A_1, A_2 are diagonal, then our SDP problem is actually an LP!

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SDP Application – Preliminaries: Rank of a Matrix

- Assume matrix A with dimensions $M \times N = 5 \times 3$
- $0 \leq \text{rank}(A) \leq \min(M, N)$
- If all rows and columns are linearly independent, then $\text{rank}(A) = \min(M, N)$
  - If all rows and columns are linearly independent, how much is $\text{rank}(A)$, if A has dimensions 5×3?

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- It holds: $\text{rank}(AB) \leq \min(\text{rank}(A), \text{rank}(B))$
- B is a vector with dimension $N \times 1$
  - How much is $\text{rank}(B)$?
  - How much is $\text{rank}(AB)$?

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- B is a vector with dimension $N \times 1$
  - How much is $\text{rank}(B)$?
  - How much is $\text{rank}(AB)$?
- $W = XX^T$, where X is a vector. How much is $\text{rank}(W)$?

Convex relaxations for AC-OPF

Disclaimer: *Illustrative example. Current SDP solvers do not use complex numbers yet. But hopefully soon! See Pascal's lecture tomorrow!*

$$
\begin{aligned}
S_1 &= V_1 Y_{\text{bus}}^* V^* \\
&= V_1 (Y_{11} V_1 + Y_{12} V_2)^* \\
&= Y_{11}^* V_1 V_1^* + Y_{12}^* V_1 V_2^*
\end{aligned}
$$

$$S_2 = Y_{22}^* V_2 V_2^* + Y_{21}^* V_2 V_1^*$$

- I define: $W = VV^H = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \begin{bmatrix} V_1^* & V_2^* \end{bmatrix} = \begin{bmatrix} V_1 V_1^* & V_1 V_2^* \\ V_2 V_1^* & V_2 V_2^* \end{bmatrix}$

It holds:

EXACT: $W \succeq 0$

$\text{rank(W)} = 1$

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It holds:

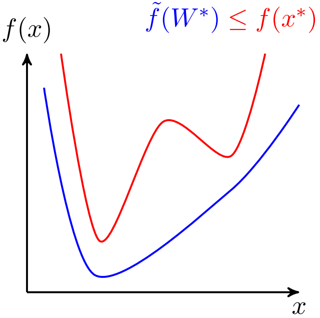
EXACT: $W \succeq 0$

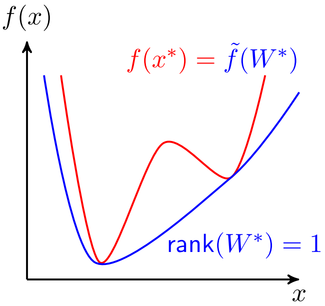
rank(W) $= 1$

RELAX: $W \succeq 0$

~~rank(W) $= 1$~~

Convex relaxations with SDP





EXACT: $W = VV^T$

$\Downarrow$

RELAX: $W \succeq 0$

~~rank$(W) = 1$~~

- For the objective functions, it holds EXACT $\geq$ RELAX
- The RELAX problem is an SDP problem!
- If W^* happens also to be rank-1, then EXACT $=$ RELAX!

Convex relaxations for AC-OPF

- The power is a quadratic function of the voltages, i.e. a function of V_i^2 and $V_i V_j$

Convex relaxations for AC-OPF

- The power is a quadratic function of the voltages, i.e. a function of V_i^2 and V_iV_j
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- We drop the rank-1 constraint $\rightarrow$ problem is now convex!

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  If yes $\rightarrow$ global optimum for the AC-OPF

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- If $\text{rank}(W_{\text{opt}}) \neq 1 \rightarrow$ infeasible, i.e. W_{opt} has no physical meaning
- It has been shown that in most power systems the obtained W_{opt} is rank-1
- Still an open research topic!

Practical application of AC-OPF: Some key points

- Complex numbers (e.g. voltage) in rectangular coordinates
- We split in real and imaginary part
- Taking advantage of the multiplicity properties of the trace operator
- $eig(W)$ to check the rank of W
- Schur's complement to transform polynomial constraints to semidefinite constraints

Splitting in real and imaginary part

$$\begin{bmatrix} V_1^r + jV_1^i \\ V_2^r + jV_2^i \\ V_3^r + jV_3^i \end{bmatrix} \Rightarrow \begin{bmatrix} V_1^r \\ V_2^r \\ V_3^r \\ V_1^i \\ V_2^i \\ V_3^i \end{bmatrix} \qquad W = XX^T = \begin{bmatrix} V_1^r \\ V_2^r \\ V_3^r \\ V_1^i \\ V_2^i \\ V_3^i \end{bmatrix} \begin{bmatrix} V_1^r & V_2^r & V_3^r & V_1^i & V_2^i & V_3^i \end{bmatrix}$$

Matrix W

We introduce the variable transformation of complex bus voltages V:

$$X := [\mathrm{Re}\{V\}\ \mathrm{Im}\{V\}]^T$$

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$$X := [\mathrm{Re}\{V\}\ \mathrm{Im}\{V\}]^T$$

The $2n_{\text{bus}}$ - dimensional vector X is transformed to a $2n_{\text{bus}} \times 2n_{\text{bus}}$ - dimensional matrix W

$$W = XX^T = \begin{bmatrix} V_1^r V_1^r & V_1^r V_2^r & \cdots & V_1^r V_n^r & V_1^r V_1^i & V_1^r V_2^i & \cdots & V_1^r V_n^i \\ V_2^r V_1^r & V_2^r V_2^r & \cdots & V_2^r V_n^r & V_2^r V_1^i & V_2^r V_2^i & \cdots & V_2^r V_n^i \\ \vdots & & \ddots & \vdots & \vdots & & \ddots & \vdots \\ V_n^r V_1^r & \cdots & \cdots & V_n^r V_n^r & V_n^r V_1^i & \cdots & \cdots & V_n^r V_n^i \\ V_1^r V_1^i & V_1^r V_2^i & \cdots & V_1^r V_n^i & V_1^i V_1^i & V_1^i V_2^i & \cdots & V_1^i V_n^i \\ V_2^r V_1^i & V_2^r V_2^i & \cdots & V_2^r V_n^i & V_2^i V_1^i & V_2^i V_2^i & \cdots & V_2^i V_n^i \\ \vdots & & \ddots & \vdots & \vdots & & \ddots & \vdots \\ V_n^r V_1^i & \cdots & \cdots & V_n^r V_n^i & V_n^i V_1^i & \cdots & \cdots & V_n^i V_n^i \end{bmatrix}$$

Properties of the trace operator

- trace(A) = sum of diagonal elements of A
- the trace is invariant under cyclic permutations:
  trace(ABC)=trace(BCA)=trace(CAB)
- Attention! trace(ABC) $\neq$ trace(BAC)

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- Example

$$tr(\mathbf{V}^T\mathbf{Y}\mathbf{V}) = tr\left(\begin{bmatrix}V_1 & V_2\end{bmatrix}\begin{bmatrix}Y_{11} & Y_{12}\\ Y_{21} & Y_{22}\end{bmatrix}\begin{bmatrix}V_1\\ V_2\end{bmatrix}\right)$$

vs

$$tr(\mathbf{Y}\mathbf{V}\mathbf{V}^T) = tr\left(\begin{bmatrix}Y_{11} & Y_{12}\\ Y_{21} & Y_{22}\end{bmatrix}\begin{bmatrix}V_1\\ V_2\end{bmatrix}\begin{bmatrix}V_1 & V_2\end{bmatrix}\right)$$

How much is the trace in each case?

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How much is the trace in each case?

What do you observe?

Note: the expression on the left is not the power flow equation. The actual expression is $diag(V)Y^*V^*$. But it gives some intuition for the next slide!

Convex Relaxation of AC-OPF

…for each node k and line lm:

Convex Relaxation of AC-OPF

...for each node k and line lm:

Minimize Generation Cost $\sum_{k\in\mathcal{G}} \{c_{k2}(\text{Tr}\{\mathbf{Y}_k W\} + P_{D_k})^2 + c_{k1}(\text{Tr}\{\mathbf{Y}_k W\} + P_{D_k}) + c_{k0}\}$

Matrices $\mathbf{Y}_k$, $\bar{\mathbf{Y}}_k$ and $\mathbf{Y}_{lm}$ are auxiliary variables resulting from the admittance matrix Y of the system.

Convex Relaxation of AC-OPF

...for each node k and line lm:

Minimize Generation Cost	$\sum_{k\in\mathcal{G}}\{c_{k2}(\mathrm{Tr}\{\mathbf{Y}_k W\}+P_{D_k})^2 + c_{k1}(\mathrm{Tr}\{\mathbf{Y}_k W\}+P_{D_k})+c_{k0}\}$
s. t. Active Power Balance	$P_k^{\text{min}} \leq \mathrm{Tr}\{\mathbf{Y}_k W\} \leq P_k^{\text{max}}$
Reactive Power Balance	$Q_k^{\text{min}} \leq \mathrm{Tr}\{\bar{\mathbf{Y}}_k W\} \leq Q_k^{\text{max}}$
Bus Voltages	$(V_k^{\text{min}})^2 \leq \mathrm{Tr}\{M_k W\} \leq (V_k^{\text{max}})^2$
Active Branch Flow	$-P_{lm}^{\text{max}} \leq \mathrm{Tr}\{\mathbf{Y}_{lm} W\} \leq P_{lm}^{\text{max}}$
Apparent Branch Flow	$\mathrm{Tr}\{\mathbf{Y}_{lm} W\}^2 + \mathrm{Tr}\{\bar{\mathbf{Y}}_{lm} W\}^2 \leq S_{lm}^{\text{max}}$

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Bus Voltages $\quad (V_k^{\text{min}})^2 \leq \text{Tr}\{M_k W\} \leq (V_k^{\text{max}})^2$

Active Branch Flow $\quad -P_{lm}^{\text{max}} \leq \text{Tr}\{\mathbf{Y}_{lm} W\} \leq P_{lm}^{\text{max}}$

Apparent Branch Flow $\quad \text{Tr}\{\mathbf{Y}_{lm} W\}^2 + \text{Tr}\{\bar{\mathbf{Y}}_{lm} W\}^2 \leq S_{lm}^{\text{max}}$

Decomposition $\quad W = \underbrace{[\text{Re}\{V\}\ \text{Im}\{V\}]^T}_{X} \underbrace{[\text{Re}\{V\}\ \text{Im}\{V\}]}_{X^T}$

Matrices $\mathbf{Y}_k$, $\bar{\mathbf{Y}}_k$ and $\mathbf{Y}_{lm}$ are auxiliary variables resulting from the admittance matrix Y of the system.

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Apparent Branch Flow	$\text{Tr}\{\mathbf{Y}_{lm} W\}^2 + \text{Tr}\{\bar{\mathbf{Y}}_{lm} W\}^2 \leq S_{lm}^{\text{max}}$
Semi-Definiteness of W	$W \succeq 0$
Rank Constraint	$\text{rank}(W) = 1$

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Semi-Definiteness of W	$W \succeq 0$
Rank Constraint	~~$\text{rank}(W)=1$~~ $\Rightarrow$ Convex Relaxation

Matrices $\mathbf{Y}_k$, $\bar{\mathbf{Y}}_k$ and $\mathbf{Y}_{lm}$ are auxiliary variables resulting from the admittance matrix Y of the system.

Zero relaxation gap: Recovering a feasible solution

- If rank(W)=1, then we have zero relaxation gap!

[1] Javad Lavaei and Steven H Low. "Zero duality gap in optimal power flow problem". In: *IEEE Transactions on Power Systems* 27.1 (2012), pp. 92–107

Zero relaxation gap: Recovering a feasible solution

- If rank(W)=1, then we have zero relaxation gap!

For the AC-OPF problem, Lavaei and Low [4]show

- rank$(W) = 1$ or 2 solution to original OPF problem can be recovered
- rank$(W) \geq 3$ solution to original OPF problem cannot be recovered

[1] Javad Lavaei and Steven H Low. "Zero duality gap in optimal power flow problem". In: *IEEE Transactions on Power Systems* 27.1 (2012), pp. 92–107

What if rank(W) = 2?

If rank$(W) = 2$, then apply eigendecomposition according to Molzahn et al.[5]:

$$W_{\text{opt}} = \rho_1 E_1 E_1^T + \rho_2 E_2 E_2^T$$
$$X_{\text{opt}} = \sqrt{\rho_1^{\text{opt}}} E_1^{\text{opt}} + \sqrt{\rho_2^{\text{opt}}} E_2^{\text{opt}}$$

The terms ρ_1, ρ_2 denote the first and second largest absolute eigenvalue of W and E_1 and E_2 the corresponding eigenvectors.

[2] Daniel K Molzahn et al. "Implementation of a large-scale optimal power flow solver based on semidefinite programming". In: *IEEE Transactions on Power Systems* 28.4 (2013), pp. 3987–3998

How to practically check the rank of a matrix?

1. Find matrix W
2. Get the eigenvalues of W
3. Check the ratio of the second largest ρ_2 to the third largest eigenvalue ρ_3

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How to practically check the rank of a matrix?

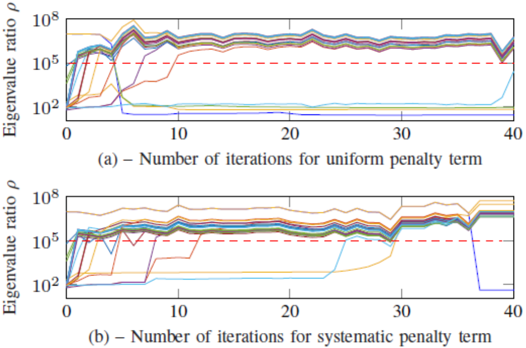
1. Find matrix W
2. Get the eigenvalues of W
3. Check the ratio of the second largest ρ_2 to the third largest eigenvalue ρ_3
4. If $\frac{\rho_2}{\rho_3} > 10^5$ then matrix W can be considered rank-2 $\rightarrow$ We can recover a feasible solution $\rightarrow$ We found the global optimum!

Note: This is a heuristic criterion for convex relaxations of AC-OPF problems specifically

[2] Daniel K Molzahn et al. "Implementation of a large-scale optimal power flow solver based on semidefinite programming". In: *IEEE Transactions on Power Systems* 28.4 (2013), pp. 3987–3998

Towards Rank-1 solutions

- Penalty factor on power lossses to achieve zero relaxation gap
- Example



A. Venzke, S. Chatzivasileiadis. Convex Relaxations of Security Constrained AC Optimal Power Flow under Uncertainty. In 20th Power Systems Computation Conference, Dublin, Ireland, pages 1-7, June 2018.

Wrap-up

- The SDP is a generalization of the LP
- The main difference between the formulation of the SDP and the LP, is that the SDP requires the variables to form a positive semidefinite matrix, while the LP requires all variables to be larger than zero.
- The SDP formulation allows for more "freedom" in the variables.
- SDP can model quadratic constraints.

Wrap-up

- The SDP is a generalization of the LP
- The main difference between the formulation of the SDP and the LP, is that the SDP requires the variables to form a positive semidefinite matrix, while the LP requires all variables to be larger than zero.
- The SDP formulation allows for more "freedom" in the variables.
- SDP can model quadratic constraints.

Convex relaxations for AC-OPF

- Transform quadratic equations to SDP expressions
- Convex relaxations of AC-OPF can recover the global optimun, but under conditions! (e.g. $\text{rank}(W) = 1$)
- Achieving zero relaxation gap: major research topic
  - SDP is not the only solution
  - to be continued... Pascal tomorrow :)

Thank you!

spchatz@elektro.dtu.dk
www.chatziva.com

Further reading
Course material: http://www.chatziva.com/teaching.html
Publications: http://www.chatziva.com/publications.html

Appendix

Auxiliary Variables

A power grid consists of $\mathcal{N}$ buses and $\mathcal{L}$ lines. The set of generator buses is denoted with $\mathcal{G}$. The following auxiliary variables are introduced for each bus $k \in \mathcal{N}$ and line $(l,m) \in \mathcal{L}$:

$$
\begin{aligned}
Y_k &:= e_k e_k^T Y \\
Y_{lm} &:= (\bar{y}_{lm} + y_{lm}) e_l e_l^T - (y_{lm}) e_l e_m^T \\
\mathbf{Y}_k &:= \frac{1}{2} \begin{bmatrix} \mathrm{Re}\{Y_k + Y_k^T\} & \mathrm{Im}\{Y_k^T - Y_k\} \\ \mathrm{Im}\{Y_k - Y_k^T\} & \mathrm{Re}\{Y_k + Y_k^T\} \end{bmatrix} \\
\mathbf{Y}_{lm} &:= \frac{1}{2} \begin{bmatrix} \mathrm{Re}\{Y_{lm} + Y_{lm}^T\} & \mathrm{Im}\{Y_{lm}^T - Y_{lm}\} \\ \mathrm{Im}\{Y_{lm} - Y_{lm}^T\} & \mathrm{Re}\{Y_{lm} + Y_{lm}^T\} \end{bmatrix} \\
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M_k &:= \begin{bmatrix} e_k e_k^T & 0 \\ 0 & e_k e_k^T \end{bmatrix}
\end{aligned}
$$

The terms $\mathrm{Re}\{\cdot\}$ and $\mathrm{Im}\{\cdot\}$ denote the real and imaginary part. Matrix Y denotes the bus admittance matrix of the power grid, e_k the k-th basis vector, $\bar{y}_{lm}$ the shunt admittance of line $(l,m) \in \mathcal{L}$ and y_{lm} the series admittance.

From non-convex AC-OPF to SDP-OPF

AC-OPF[1]

obj.function	$\min c^T P_G$
AC flow	$S_G - S_L = diag(\overline{V})\overline{Y}^*_{\text{bus}}\overline{V}^*$
Line Current	$\lvert\overline{Y}_{\text{line},i\to j}\overline{V}\rvert \leq I_{line,max}$
	$\lvert\overline{Y}_{\text{line},j\to i}\overline{V}\rvert \leq I_{line,max}$
or Apparent Flow	$\lvert\overline{V}_i\overline{Y}^*_{\text{line},i\to j,\text{i-row}}\overline{V}^*\rvert \leq S_{i\to j,max}$
	$\lvert\overline{V}_j\overline{Y}^*_{\text{line},j\to i,\text{j-row}}\overline{V}^*\rvert \leq S_{j\to i,max}$
Gen. Active Power	$0 \leq P_G \leq P_{G,max}$
Gen. Reactive Power	$-Q_{G,max} \leq Q_G \leq Q_{G,max}$
Voltage Magnitude	$V_{min} \leq V \leq V_{max}$
Voltage Angle	$\delta_{min} \leq \delta \leq \delta_{max}$

[1]All shown variables are vectors or matrices. The bar above a variable denotes complex numbers. $(\cdot)^*$ denotes the complex conjugate. To simplify notation, the bar denoting a complex number is dropped in the following slides. Attention! The *current* flow constraints are defined as *vectors*, i.e. for all lines. The apparent power *line* constraints are defined *per line*.

Complex Power Injections

- Complex power balance for all buses writes:

$$S_G - S_L = diag(\bar{V})\bar{Y}_{bus}^{*}\bar{V}^{*}$$

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$\Rightarrow$ The vectors e_k are unit vectors that have a $\{1\}$ at the k-th entry. Otherwise, their entries are $\{0\}$.

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- Introducing the trace operator (sum of the diagonal elements of a matrix) and use its multiplicity property:

$$[S_G - S_L]_k = \mathsf{Tr}\{\bar{V}^T e_k e_k^T \bar{Y}_{bus}^{*}\bar{V}^{*}\} = \mathsf{Tr}\{e_k e_k^T \bar{Y}_{bus}^{*}\bar{V}^{*}\bar{V}^T\}$$

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$\Rightarrow$ AC-OPF formulation in complex variables where we could substitute $\bar{V}^{*}\bar{V}^T$ with a complex W. We split the complex formulation into real and imaginary part.

Splitting into Real and Imaginary Part

- If we have two generic complex numbers $a + jb$ and $c + jd$, then we can write their product as:

$$(a + jb)(c + jd) = ac - bd + j(ad + bc)$$

- In matrix form this can be formulated as:

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$$(a + jb)(c + jd) = ac - bd + j(ad + bc)$$

- In matrix form this can be formulated as:

$$\begin{bmatrix} \mathrm{Re} \\ \mathrm{Im} \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix}$$

Splitting into Real and Imaginary Part

- Following this logic, we can write the real part of the active power injections using $Y_k := e_k e_k^T Y$ as:

$$\mathrm{Re}\{[S_G - S_L]_k\} = \mathrm{Re}\{\bar{V}^T e_k e_k^T \bar{Y}_{bus}^* \bar{V}^*\}$$

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&= X^T \begin{bmatrix} \mathrm{Re}\{Y_k\} & -\mathrm{Im}\{Y_k\} \\ \mathrm{Im}\{Y_k\} & \mathrm{Re}\{Y_k\} \end{bmatrix} X \\
&= X^T \frac{1}{2} \begin{bmatrix} \mathrm{Re}\{Y_k + Y_k^T\} & \mathrm{Im}\{Y_k^T - Y_k\} \\ \mathrm{Im}\{Y_k - Y_k^T\} & \mathrm{Re}\{Y_k + Y_k^T\} \end{bmatrix} X \\
&= X^T \mathbf{Y}_k X = \mathsf{Tr}\{\mathbf{Y}_k X X^T\}
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&= X^T \mathbf{Y}_k X = \mathsf{Tr}\{\mathbf{Y}_k X X^T\}
\end{aligned}$$

$\Rightarrow$ This procedure can be similarly applied to yield the mathematical formulation of the reactive power injections and the active and apparent branch flows.

Schur's Complement

The objective on generation cost and the constraint on apparent line flow cannot be directly implemented in the SDP.

We can use the so-called Schur's complement to reformulate polynomial equations as semidefinite constraints.

Schur's Complement

The Schur complement is defined as follows[6]. Given a matrix $X \in S^n$ which can be partitioned in the sub-matrices A, B and C:

$$X = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \tag{1}$$

If det $A \neq 0$, the matrix

$$S = C - B^T A^{-1} B \tag{2}$$

is called the Schur complement of A in X. The following statements can be made regarding the positive semi-definiteness of the matrix X:

- $X \succ 0$ if and only if $A \succ 0$ and $S \succ 0$
- If $A \succ 0$, then $X \succeq 0$ if and only if $S \succeq 0$

[3] **boyd2004convex**

Schur's Complement

To obtain an optimization problem linear in W, the objective function is reformulated using Schur's complement:

$$\min_{W,\alpha} \sum_{k\in\mathcal{G}} \alpha_k$$

$$\begin{bmatrix} c_{k1}\mathsf{Tr}\{\mathbf{Y}_k W\} + a_k & \sqrt{c_{k2}}\mathsf{Tr}\{\mathbf{Y}_k W\} + b_k \\ \sqrt{c_{k2}}\mathsf{Tr}\{\mathbf{Y}_k W\} + b_k & -1 \end{bmatrix} \preceq 0$$

where $a_k := -\alpha_k + c_{k0} + c_{k1}P_{D_k}$ and $b_k := \sqrt{c_{k2}}P_{D_k}$. The variable α is introduced as an additional optimization variable. In addition, the apparent branch flow constraint is rewritten:

$$\begin{bmatrix} -(\overline{S}_{lm})^2 & \mathsf{Tr}\{\mathbf{Y}_{lm} W\} & \mathsf{Tr}\{\bar{\mathbf{Y}}_{lm} W\} \\ \mathsf{Tr}\{\mathbf{Y}_{lm} W\} & -1 & 0 \\ \mathsf{Tr}\{\bar{\mathbf{Y}}_{lm} W\} & 0 & -1 \end{bmatrix} \preceq 0$$

Schur's Complement

This theorem is used to prove that the semi-definite constraint is equal to the quadratic constraint. The matrix X corresponds to

$$X = \begin{bmatrix} \overline{S}_{lm}^2 & \mathrm{Tr}\{\mathbf{Y}_{lm}W\} & \mathrm{Tr}\{\bar{\mathbf{Y}}_{lm}W\} \\ \mathrm{Tr}\{\mathbf{Y}_{lm}W\} & 1 & 0 \\ \mathrm{Tr}\{\bar{\mathbf{Y}}_{lm}W\} & 0 & 1 \end{bmatrix} \succeq 0$$

Applying Schur complement a first time, defining

$$A = \overline{S}_{lm}^2 \qquad B = \begin{bmatrix} \mathrm{Tr}\{\mathbf{Y}_{lm}W\} & \mathrm{Tr}\{\bar{\mathbf{Y}}_{lm}W\} \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

yields the following result:

Schur's Complement

$$
\begin{aligned}
S_1 &= C - B^T A^{-1} B \\
&= \begin{bmatrix} 1 - \dfrac{\text{Tr}\{\mathbf{Y}_{lm} W\}^2}{\overline{S}_{lm}^2} & \dfrac{\text{Tr}\{\mathbf{Y}_{lm} W\}\text{Tr}\{\bar{\mathbf{Y}}_{lm} W\}}{\overline{S}_{lm}^2} \\ \dfrac{\text{Tr}\{\mathbf{Y}_{lm} W\}\text{Tr}\{\bar{\mathbf{Y}}_{lm} W\}}{\overline{S}_{lm}^2} & 1 - \dfrac{\text{Tr}\{\bar{\mathbf{Y}}_{lm} W\}^2}{\overline{S}_{lm}^2} \end{bmatrix} \succeq 0
\end{aligned}
$$

If Schur complement is applied a second time, the result is the initial quadratic constraint:

$$
S_2 = \overline{S}_{lm}^2 - \text{Tr}\{\mathbf{Y}_{lm} W\}^2 - \text{Tr}\{\bar{\mathbf{Y}}_{lm} W\}^2 \geq 0
$$

Hence, the proof is completed. In the context of semi-definite programming, Schur complement is a powerful tool, which can be used to transform polynomial constraints into semi-definite constraints.