Optimal Power Flow (DC-OPF and AC-OPF)

DTU Summer School 2018

Spyros Chatzivasileiadis

DTU Electrical Engineering
Department of Electrical Engineering

What is optimal power flow?

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- In its most realistic form, the OPF is a non-linear, non-convex problem, which includes both binary and continuous variables.

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- **Disclaimer:**
  - Realistic OPF implementations include thousands of variables and constraints
  - Here we focus on the most "fundamental" formulations of OPF

Outline

- Economic Dispatch
- DC-OPF
- AC-OPF

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- Economic Dispatch
  - used in Power Pools and Power Exchanges
  - Supply must meet demand
  - Generator limits

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  - considers line limits and the power flows! (linearized)
  - only active power; no losses

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  - only active power; no losses
- AC-OPF
  - "Security-Constrained AC-OPF: ultimate goal for market software"[1]
  - not only markets: minimize losses, optimize voltage profile, and others
  - full AC power flow equations
  - active and reactive power flow, current, voltage, losses

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Economic Dispatch

Find the cheapest generators that can cover the total demand! How?

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subject to:

$$P_{G_i}^{min} \leq P_{G_i} \leq P_{G_i}^{max}$$
$$\sum_i P_{G_i} = P_D$$

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- The Economic Dispatch does not consider any network flows or network constraints!
- We assume a copperplate network, i.e. a lossless and unrestricted flow of electricity from A to B.

Can we solve the economic dispatch problem without using an optimization solver?

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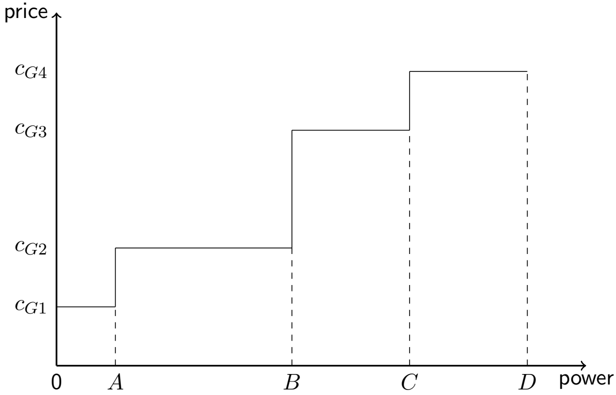
Yes! With the help of the merit order curve.

The Merit-Order Curve



$$A = P_{G1}^{max}$$

$$B = A + P_{G2}^{max}$$

$$C = B + P_{G3}^{max}$$

$$D = C + P_{G4}^{max}$$

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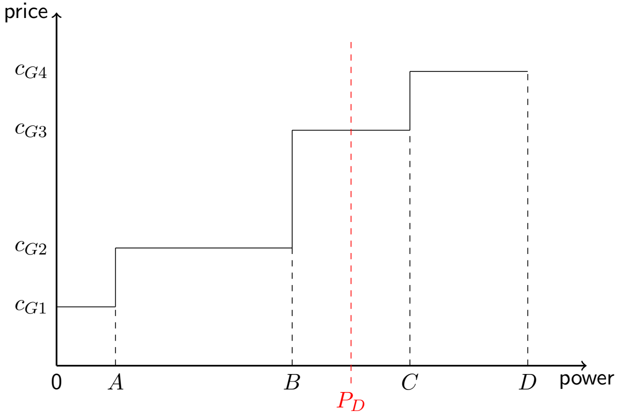

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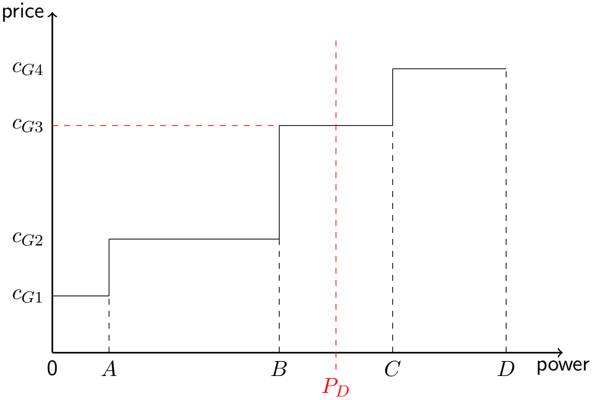
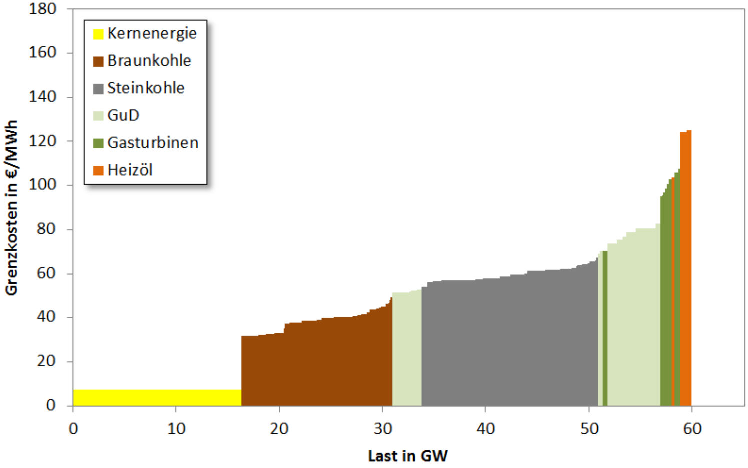


- c_{G3} is the system marginal price
- G1 and G2 fully dispatched
- G4 not dispatched
- G3 partially dispatched: "marginal generator"

The Merit-Order Curve: An Example



Merit-Order of the German conventional generation in 2008

Source: Forschungsstelle für Energiewirtschaft e. V.

Line Congestion and Marginal Generators

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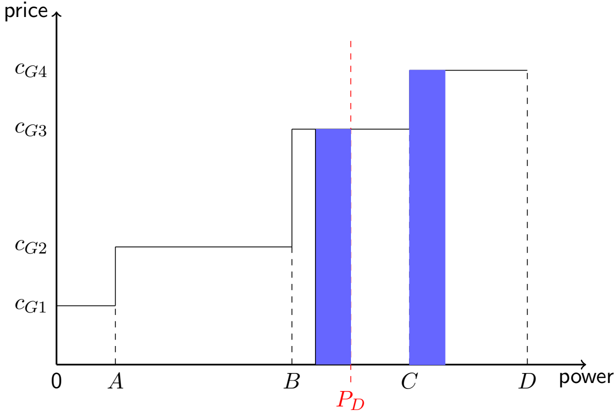
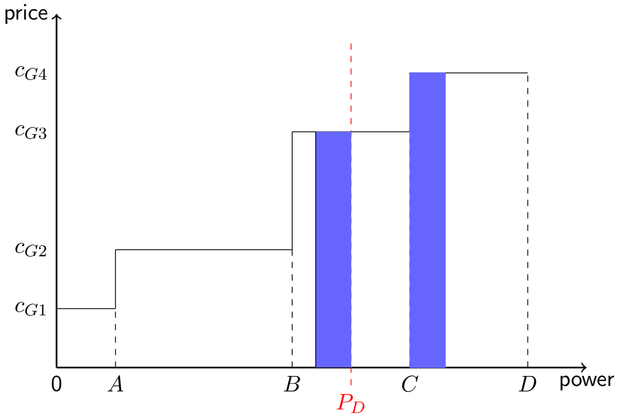


- Although G3 has enough capacity, it cannot produce enough to cover the demand due to line congestion
- Instead G4, a more expensive gen, must produce the missing power

Line Congestion and Marginal Generators



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- In a DC-OPF context, there is no longer a single system marginal price (we will observe different nodal prices in different nodes)

DC-OPF vs Economic Dispatch

What is the difference?

DC-OPF vs Economic Dispatch

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DC-OPF includes the line flow constraints!

So how do I do that?

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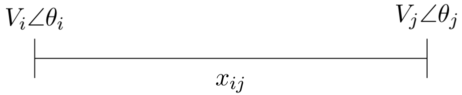
Linearized power flow equations



simplified model of the line

$$P_{ij} = \frac{V_i V_j}{x_{ij}} \sin(\theta_i - \theta_j) \quad \xrightarrow{\text{linearize}} \quad P_{ij} = \frac{1}{x_{ij}} (\theta_i - \theta_j)$$

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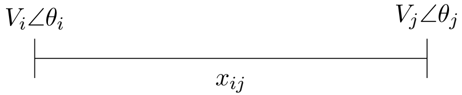

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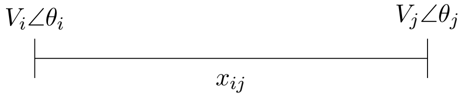
2. How do I include the line flow constraint in the DC-OPF?

Linearized power flow equations



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1. What are my assumptions for linearizing the power flow?
   $V_i = V_j = 1$p.u and $sin(\theta_i - \theta_j) \approx \theta_i - \theta_j$

2. How do I include the line flow constraint in the DC-OPF?

DC-OPF

$$\min \sum_i c_i P_{G_i}$$

subject to:

$$P_{G_i}^{min} \leq P_{G_i} \leq P_{G_i}^{max}$$

$$\left| \frac{1}{x_{ij}} (\theta_i - \theta_j) \right| \leq P_{ij,max}$$

- The line flow constraints must include both directions!

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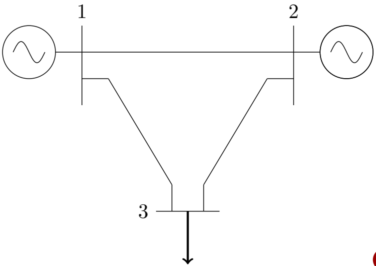
$$\mathbf{B} \cdot \theta = \mathbf{P_G} - \mathbf{P_D}$$

- The line flow constraints must include both directions!
- The DC-OPF with the standard power flow equations contains both the power generation $\mathbf{P_G}$ and the voltage angles θ in the vector of the optimization variables.

Exercise



$c_{G1} = 60$ \$/MWh, $c_{G2} = 120$ \$/MWh

$P_{load} = 150$ MW

$P_{G1}^{max} = 100$ MW, $P_{G2}^{max} = 200$ MW

$X_{12} = 0.1$ pu, $X_{13} = 0.3$ pu, $X_{23} = 0.1$ pu,
BaseMVA = 100 MVA

$P_{13}^{max} = 40\ MW$ (line limit)

1. What are the optimization variables? Form the optimization vector
2. Formulate the objective function
3. Formulate the constraints

DC-OPF in Matlab

linprog

Solve linear programming problems

Linear programming solver

Finds the minimum of a problem specified by

$$\min_x f^T x \text{ such that } \begin{cases} A \cdot x \leq b, \\ Aeq \cdot x = beq, \\ lb \leq x \leq ub. \end{cases}$$

f, *x*, *b*, *beq*, *lb*, and *ub* are vectors, and *A* and *Aeq* are matrices.

Syntax

```
x = linprog(f,A,b)
x = linprog(f,A,b,Aeq,beq)
x = linprog(f,A,b,Aeq,beq,lb,ub)
x = linprog(f,A,b,Aeq,beq,lb,ub,x0)
x = linprog(f,A,b,Aeq,beq,lb,ub,x0,options)
x = linprog(problem)
[x,fval] = linprog( __ )
[x,fval,exitflag,output] = linprog( __ )
[x,fval,exitflag,output,lambda] = linprog( __ )
```

How would you transfer your problem formulation to Matlab?

How do you calculate the nodal prices?

Discussion Points

- $\sin\delta \approx \delta$
  - δ is in rad!

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  - θ is in rad, $\Rightarrow$ dimensionless
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- Bus Admittance Matrix $\mathbf{B}$ in DC-OPF
  - $b_{ij} = \frac{1}{x_{ij}} \Rightarrow$ positive
  - all off-diagonal elements are non-positive (zero or negative)
  - all diagonal elements are positive
  - AC-OPF: This differs from the case where $z_{ij} = r_{ij} + jx_{ij}$. In that case, it is $y_{ij} = g_{ij} + jb_{ij}$ with b_{ij} is negative.

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- If the DC-OPF does not converge, check that the admittance matrix B is correct!

Some additional points...

- Nodal prices

  In a market context, the nodal prices are:

  - the lagrangian multipliers of the equality constraints $B\theta = P$
  - of a DC-OPF (at the moment)
  - with objective function the minimization of costs

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- Power Transfer Distribution Factors (PTDFs)
  - PTDFs are linear sensitivies that relate the line flows to the power injections
  - the DC-OPF can be formulated with respect to PTDFs
  - PTDFs eliminate the need of θ as optimization variable
  - In the zonal pricing in Europe PTDFs are used to model the flows between the zones

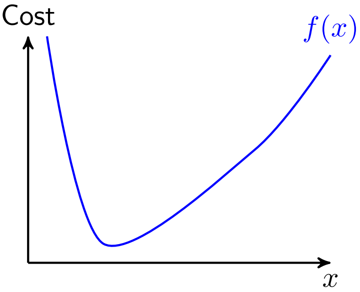
4-slide “break”

DC-OPF: linear program $=$ convex

AC-OPF: non-linear non-convex problem in its original form
$\Rightarrow$ recent efforts to convexify the problem
Why?

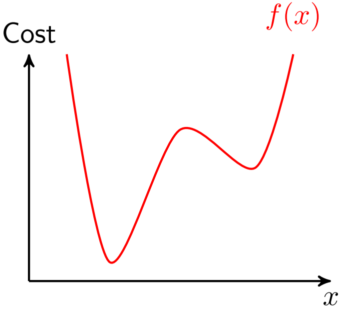
Convex vs. Non-convex Problem

Convex Problem



One global minimum

Non-convex problem

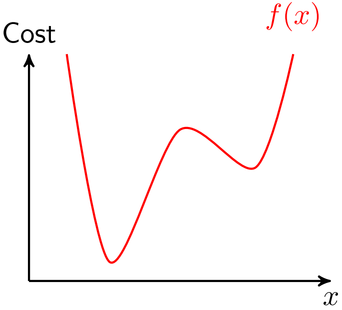


Several local minima

Several local minima: So what?

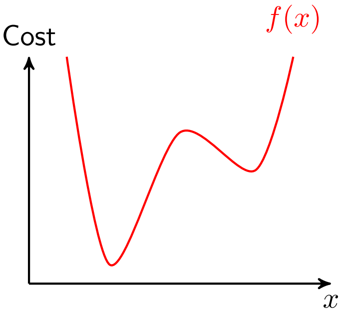
Example: Optimal Power Flow Problem

- Assume that the difference in the cost function of a local minimum versus a global minimum is 1%
- The total electric energy cost in the US is $\approx$ 400 Billion$/year
- 1% amounts to 4 billion US$ in economic losses per year
- Convexifying AC-OPF
  1. guarantees that we find a global minimum or
  2. at least determines how far we are from the global minimum



Convexifying the Optimal Power Flow problem (OPF)

- Convex relaxations transform the OPF to a convex Semi-Definite Program (SDP)



Convex Relaxation

[2] Javad Lavaei and Steven H Low. "Zero duality gap in optimal power flow problem". In: *IEEE Transactions on Power Systems* 27.1 (2012), pp. 92–107

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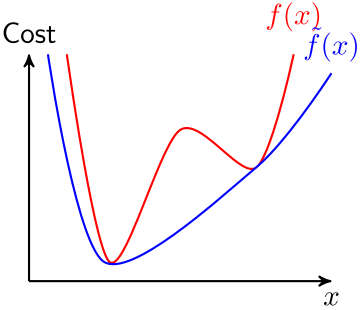

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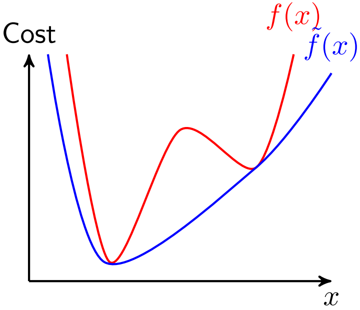
Convexifying the Optimal Power Flow problem (OPF)

- Convex relaxations transform the OPF to a convex Semi-Definite Program (SDP)
- Under certain conditions, the obtained solution is the global optimum to the original OPF problem[2]



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Break is over...
More in 1 hour and in Pascal's talk tomorrow!

Be patient :)

AC-OPF

- Minimize

- subject to:

AC-OPF

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Costs, Line Losses, other?

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AC Power Flow equations

Line Flow Constraints

Generator Active Power Limits

Generator Reactive Power Limits

Voltage Magnitude Limits

(Voltage Angle limits to improve solvability)

(maybe other equipment constraints)

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AC Power Flow equations

Line Flow Constraints

- Line Current Limits
- Apparent Power Flow limits
- Active Power Flow limits

Generator Active Power Limits

Generator Reactive Power Limits

Voltage Magnitude Limits

(Voltage Angle limits to improve solvability)

(maybe other equipment constraints)

- Optimization vector: $[P\ Q\ V\ \theta]^T$

AC-OPF[3]

obj.function	$\min c^T P_G$
Gen. Active Power	$0 \leq P_G \leq P_{G,max}$
Gen. Reactive Power	$-Q_{G,max} \leq Q_G \leq Q_{G,max}$
Voltage Magnitude	$V_{min} \leq V \leq V_{max}$
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[3] All shown variables are vectors or matrices. The bar above a variable denotes complex numbers. $(\cdot)^*$ denotes the complex conjugate. To simplify notation, the bar denoting a complex number is dropped in the following slides. Attention! The *current* flow constraints are defined as *vectors*, i.e. for all lines. The apparent power *line* constraints are defined *per line*.

AC-OPF[3]

obj.function	$\min c^T P_G$
Line Current	$\lvert \overline{Y}_{\text{line},i\rightarrow j}\overline{V}\rvert \leq I_{line,max}$
	$\lvert \overline{Y}_{\text{line},j\rightarrow i}\overline{V}\rvert \leq I_{line,max}$
or Apparent Flow	$\lvert \overline{V}_i \overline{Y}^*_{\text{line},i\rightarrow j,\text{i-row}}\overline{V}^*\rvert \leq S_{i\rightarrow j,max}$
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AC-OPF[3]

obj.function	$\min c^T P_G$
AC flow	$S_G - S_L = diag(\overline{V})\overline{Y}^*_{\text{bus}}\overline{V}^*$
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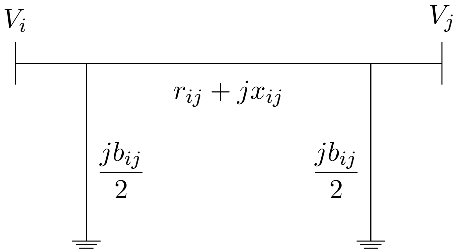
Current flow along a line



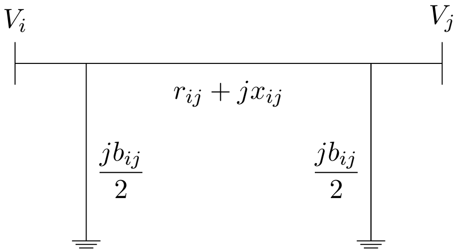
π-model of the line

It is:

$$y_{ij} = \frac{1}{r_{ij} + jx_{ij}}$$

$y_{sh,i} = j\frac{b_{ij}}{2} +$ other shunt elements connected to that bus

Current flow along a line



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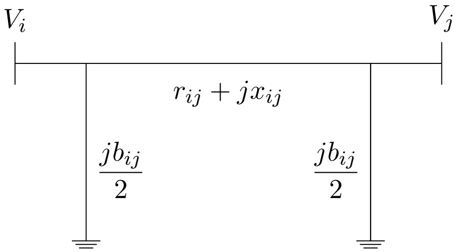
$$i \rightarrow j: \quad I_{i\rightarrow j} = y_{sh,i}V_i + y_{ij}(V_i - V_j) \Rightarrow I_{i\rightarrow j} = \begin{bmatrix} y_{sh,i} + y_{ij} & -y_{ij} \end{bmatrix} \begin{bmatrix} V_i \\ V_j \end{bmatrix}$$

Current flow along a line



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Line Admittance Matrix Y_{line}

- $\mathbf{Y}_{\text{line}}$ is an $L \times N$ matrix, where L is the number of lines and N is the number of nodes

- if row k corresponds to line $i-j$:
  - $Y_{\text{line},ki} = y_{sh,i} + y_{ij}$
  - $Y_{\text{line},kj} = -y_{ij}$

- $y_{ij} = \dfrac{1}{r_{ij} + jx_{ij}}$ is the admittance of line ij
- $y_{sh,i}$ is the shunt capacitance $jb_{ij}/2$ of the π-model of the line
- We must create two Y_{line} matrices. One for $i \to j$ and one for $j \to i$

Bus Admittance Matrix Y_{bus}

$$S_i = V_i I_i^*$$
$$I_i = \sum_k I_{ik}, \text{ where } k \text{ are all the buses connected to bus } i$$

Example: Assume there is a line between nodes $i - m$, and $i - n$. It is:

$$\begin{aligned} I_i &= I_{im} + I_{in} \\ &= (y_{sh,i}^{i\to m} + y_{im})V_i - y_{im}V_m + (y_{sh,i}^{i\to n} + y_{in})V_i - y_{in}V_n \\ &= (y_{sh,i}^{i\to m} + y_{im} + y_{sh,i}^{i\to n} + y_{in})V_i - y_{im}V_m - y_{in}V_n \end{aligned}$$

$$I_i = [\underbrace{y_{sh,im} + y_{im} + y_{sh,in} + y_{in}}_{Y_{\text{bus},ii}} \; \underbrace{-y_{im}}_{Y_{\text{bus},im}} \; \underbrace{-y_{in}}_{Y_{\text{bus},in}}][V_i \; V_m \; V_n]^T$$

Bus Admittance Matrix Y_{bus}

- $\mathbf{Y}_{\text{bus}}$ is an $N \times N$ matrix, where N is the number of nodes
- diagonal elements: $Y_{\text{bus},ii} = y_{sh,i} + \sum_k y_{ik}$, where k are all the buses connected to bus i
- off-diagonal elements:
  - $Y_{\text{bus},ij} = -y_{ij}$ if nodes i and j are connected by a line[4]
  - $Y_{\text{bus},ij} = 0$ if nodes i and j are not connected
- $y_{ij} = \dfrac{1}{r_{ij} + jx_{ij}}$ is the admittance of line ij
- $y_{sh,i}$ are all shunt elements connected to bus i, including the shunt capacitance of the π-model of the line

[4] If there are more than one lines connecting the same nodes, then they must all be added to $Y_{\text{bus},ij}, Y_{\text{bus},ii}, Y_{\text{bus},jj}$.

AC Power Flow Equations

$$S_i = V_i I_i^*$$
$$= V_i Y_{\text{bus}}^* V^*$$

For all buses $S = [S_1 \dots S_N]^T$:

$$S_{\text{gen}} - S_{\text{load}} = diag(V) Y_{\text{bus}}^* V^*$$

Further reading

- Resources about AC-OPF from the US Federal Energy Regulatory Commission (FERC)

  `https://www.ferc.gov/industries/electric/indus-act/market-planning/opf-papers.asp`

- Overview paper on Economic Dispatch and DC-OPF:

  R.D. Christie, B. F. Wollenberg, I. Wangesteen, Transmission Management in the Deregulated Environment, Proceedings of the IEEE, vol. 88, no. 2, February 2000

- DTU Lecture slides: Optimization in modern power systems

  `http://www.chatziva.com/teaching/2017/31765.html`

- Line Congestion, Nodal Prices, and Marginal Generators

  S. Chatzivasileiadis, T. Krause, and G. Andersson. HVDC line placement for maximizing social welfare - an analytical approach. In IEEE Powertech 2013, pages 1 -6, June 2013.

DC-OPF

- market clearing uses DC-OPF (at the moment)
- convex
- can solve fast; can be applied in very large problems

but

- only active power flow
- no losses and no voltage limits

DC approximations more suitable for transmission systems (not distribution)

AC-OPF

- primarily used for optimization of operation and control actions
- future: use in markets
- full AC power flow equations

but

- non-convex (in its original form) $\rightarrow$ no guarantee that we find the global optimum
- computationally expensive and intractable for very large systems

efforts to decrease computation time and increase system size

Thank you!

spchatz@elektro.dtu.dk
www.chatziva.com

Appendix

From AC to DC Power Flow Equations

- The power flow along a line is:

$$S_{ij} = V_i I_{ij}^* = V_i (y_{sh,i}^* V_i^* + y_{ij}^* (V_i^* - V_j^*))$$

- Assume a negligible shunt conductance: $g_{sh,ij} = 0 \Rightarrow y_{sh,i} = jb_{sh,i}$.
- Given that $R << X$ in transmission systems, for the DC power flow we assume that $z_{ij} = r_{ij} + jx_{ij} \approx jx_{ij}$. Then $y_{ij} = -j\frac{1}{x_{ij}}$.
- Assume: $V_i = V_i \angle 0$ and $V_j = V_j \angle \delta$, with $\delta = \theta_j - \theta_i$.

$$\begin{aligned} I_{ij}^* &= -jb_{sh,i} V_i + j\frac{1}{x_{ij}} (V_i - (V_j \cos\delta - jV_j \sin\delta)) \\ &= -jb_{sh,i} V_i + j\frac{1}{x_{ij}} V_i - j\frac{1}{x_{ij}} V_j \cos\delta - \frac{1}{x_{ij}} V_j \sin\delta \end{aligned}$$

From AC to DC Power Flow Equations (cont.)

- Since V_i is a real number, it is:

$$P_{ij} = \Re\{S_{ij}\} = V_i \Re\{I^*_{ij}\} = -\frac{1}{x_{ij}} V_i V_j \sin\delta$$

- With $\delta = \theta_j - \theta_i$, it is:

$$P_{ij} = \frac{1}{x_{ij}} V_i V_j \sin(\theta_i - \theta_j)$$

- We further make the assumptions that:
  - V_i, V_j are constant and equal to 1 p.u.
  - $\sin\theta \approx \theta$, θ must be in rad

Then

$$P_{ij} = \frac{1}{x_{ij}} (\theta_i - \theta_j)$$